6.094 Introduction to MATLAB®
January (IAP) 2009

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Introduction to Programming in MATLAB®

Lecture 3 : Solving Equations and Curve Fitting

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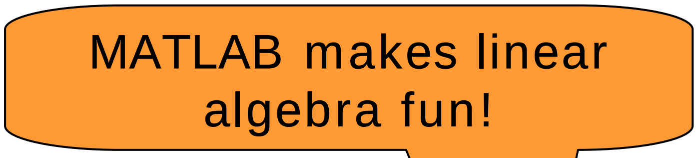
IAP 2009

Outline

(1) Linear Algebra
(2) Polynomials
(3) Optimization
(4) Differentiation/Integration
(5) Differential Equations

Systems of Linear Equations



- Given a system of linear equations
  - x+2y-3z=5
  - -3x-y+z=-8
  - x-y+z=0
- Construct matrices so the system is described by Ax=b

```
» A=[1 2 -3;-3 -1 1;1 -1 1];
» b=[5;-8;0];
```

- And solve with a single line of code!

```
» x=A\b;
```

  - x is a 3x1 vector containing the values of x, y, and z

- The **** will work with square or rectangular systems.
- Gives least squares solution for rectangular systems. Solution depends on whether the system is over or underdetermined.

More Linear Algebra

- Given a matrix
  - » `mat=[1 2 -3;-3 -1 1;1 -1 1];`
- Calculate the rank of a matrix
  - » `r=rank(mat);`
    - ➢ the number of linearly independent rows or columns
- Calculate the determinant
  - » `d=det(mat);`
    - ➢ mat must be square
    - ➢ if determinant is nonzero, matrix is invertible
- Get the matrix inverse
  - » `E=inv(mat);`
    - ➢ if an equation is of the form A*x=b with A a square matrix, x=A\b is the same as x=inv(A)*b

Matrix Decompositions

- MATLAB has built-in matrix decomposition methods

- The most common ones are
  - » `[V,D]=eig(X)`
    - ➢ Eigenvalue decomposition
  - » `[U,S,V]=svd(X)`
    - ➢ Singular value decomposition
  - » `[Q,R]=qr(X)`
    - ➢ QR decomposition

Exercise: Linear Algebra

- Solve the following systems of equations:

  - System 1:
  - $x+4y=34$
  - $-3x+y=2$

  - System 2:
  - $2x-2y=4$
  - $-x+y=3$
  - $3x+4y = 2$

Exercise: Linear Algebra

- Solve the following systems of equations:

  - System 1:
  - x+4y=34
  - -3x+y=2

```
» A=[1 4;-3 1];
» b=[34;2];
» rank(A)
» x=inv(A)*b;
```

  - System 2:
  - 2x-2y=4
  - -x+y=3
  - 3x+4y = 2

```
» A=[2 -2;-1 1;3 4];
» b=[4;3;2];
» rank(A)
```

  - rectangular matrix

```
» x1=A\b;
```

  - gives least squares solution

```
» A*x1
```

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(1) Linear Algebra

(2) Polynomials

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Polynomials

- Many functions can be well described by a high-order polynomial

- MATLAB represents a polynomials by a vector of coefficients
  - if vector P describes a polynomial
    - $ax^3 + bx^2 + cx + d$

      P(1) P(2) P(3) P(4)

- P=[1 0 -2] represents the polynomial x^2-2

- P=[2 0 0 0] represents the polynomial $2x^3$

Polynomial Operations

- P is a vector of length N+1 describing an N-th order polynomial
- To get the roots of a polynomial
  - » **r=roots(P)**
    - ➢ r is a vector of length N

- Can also get the polynomial from the roots
  - » **P=poly(r)**
    - ➢ r is a vector length N

- To evaluate a polynomial at a point
  - » **y0=polyval(P,x0)**
    - ➢ x0 is a single value; y0 is a single value

- To evaluate a polynomial at many points
  - » **y=polyval(P,x)**
    - ➢ x is a vector; y is a vector of the same size

Polynomial Fitting

- MATLAB makes it very easy to fit polynomials to data

- Given data vectors X=[-1 0 2] and Y=[0 -1 3]
  - » **p2=polyfit(X,Y,2);**
    - finds the best second order polynomial that fits the points (-1,0),(0,-1), and (2,3)
    - see **help polyfit** for more information
  - » **plot(X,Y,'o', 'MarkerSize', 10);**
  - » **hold on;**
  - » **x = linspace(-2,2,1000);**
  - » **plot(x,polyval(p2,x), 'r--');**

Exercise: Polynomial Fitting

- Evaluate x^2 over x=-4:0.1:4 and save it as y.

- Add random noise to these samples. Use **randn**. Plot the noisy signal with . markers

- fit a 2nd degree polynomial to the noisy data

- plot the fitted polynomial on the same plot, using the same x values and a red line

Exercise: Polynomial Fitting

- Evaluate x^2 over x=-4:0.1:4 and save it as y.
  - » **x=-4:0.1:4;**
  - » **y=x.^2;**

- Add random noise to these samples. Use **randn**. Plot the noisy signal with . markers
  - » **y=y+randn(size(y));**
  - » **plot(x,y,'.');**
- fit a 2nd degree polynomial to the noisy data
  - » **[p]=polyfit(x,y,2);**
- plot the fitted polynomial on the same plot, using the same x values and a red line
  - » **hold on;**
  - » **plot(x,polyval(p,x),'r')**

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Nonlinear Root Finding

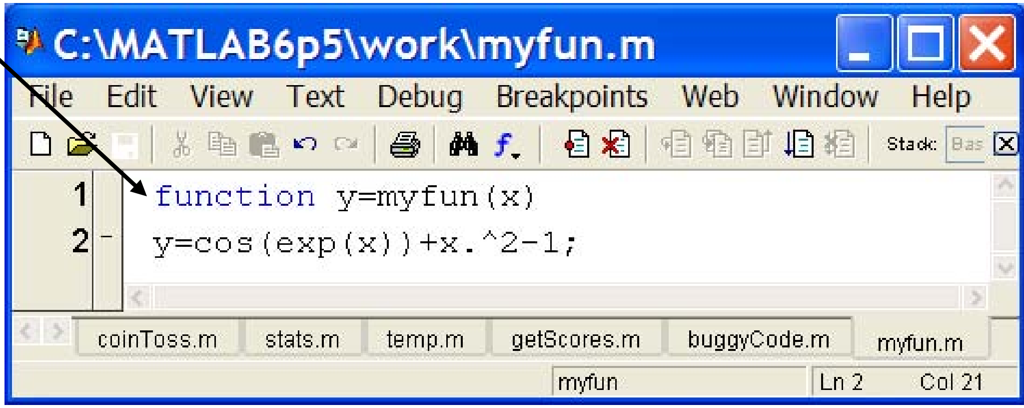
- Many real-world problems require us to solve f(x)=0
- Can use **fzero** to calculate roots for *any* arbitrary function

- **fzero** needs a function passed to it.
- We will see this more and more as we delve into solving equations.

- Make a separate function file

```
» x=fzero('myfun',1)
» x=fzero(@myfun,1)
```

  - 1 specifies a point close to the root



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Minimizing a Function

- **fminbnd**: minimizing a function over a bounded interval
  - » `x=fminbnd('myfun',-1,2);`
    - ➢ myfun takes a scalar input and returns a scalar output
    - ➢ myfun(x) will be the minimum of myfun for $-1 \leq x \leq 2$
- **fminsearch**: unconstrained interval
  - » `x=fminsearch('myfun',.5)`
    - ➢ finds the local minimum of myfun starting at x=0.5

Anonymous Functions

- You do not have to make a separate function file

- Instead, you can make an anonymous function

```
» x=fzero(@(x)(cos(exp(x))+x^2-1),  1  );
```

input   function to evaluate

```
» x=fminbnd(@(x) (cos(exp(x))+x^2-1),-1,2);
```

Optimization Toolbox

- If you are familiar with optimization methods, use the optimization toolbox

- Useful for larger, more structured optimization problems

- Sample functions (see help for more info)
  - » **`linprog`**
    - ➢ linear programming using interior point methods
  - » **`quadprog`**
    - ➢ quadratic programming solver
  - » **`fmincon`**
    - ➢ constrained nonlinear optimization

Exercise: Min-Finding

Find the minimum of the function f(x) = cos(4*x).*sin(10*x).*exp(-abs(x)) over the range –pi to pi. Use fminbnd. Is your answer really the minimum over this range?

Exercise: Min-Finding

Find the minimum of the function f(x) = cos(4*x).*sin(10*x).*exp(-abs(x)) over the range –pi to pi. Use fminbnd. Is your answer really the minimum over this range?

```
function y = myFun(x)
y=cos(4*x).*sin(10*x).*exp(-abs(x));

fminbnd('myFun', -pi, pi);
```

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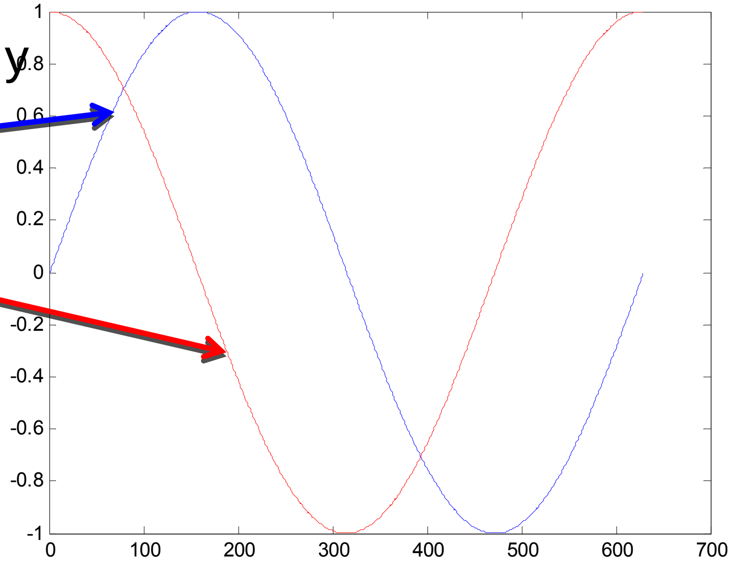
Numerical Differentiation

- MATLAB can 'differentiate' numerically
  » **x=0:0.01:2*pi;**
  » **y=sin(x);**
  » **dydx=diff(y)./diff(x);**
    - diff computes the first difference



- Can also operate on matrices
  » **mat=[1 3 5;4 8 6];**
  » **dm=diff(mat,1,2)**
    - first difference of mat along the 2nd dimension, dm=[2 2;4 -2]
    - see **help** for more details

- 2D gradient
  » **[dx,dy]=gradient(mat);**

Numerical Integration

- MATLAB contains common integration methods

- Adaptive Simpson's quadrature (input is a function)
  - » **`q=quad('derivFun',0,10);`**
    - ➢ q is the integral of the function derivFun from 0 to 10
  - » **`q2=quad(@sin,0,pi)`**
    - ➢ q2 is the integral of sin from 0 to pi
- Trapezoidal rule (input is a vector)
  - » **`x=0:0.01:pi;`**
  - » **`z=trapz(x,sin(x));`**
    - ➢ z is the integral of sin(x) from 0 to pi
  - » **`z2=trapz(x,sqrt(exp(x))./x)`**
    - ➢ z2 is the integral of $\sqrt{e^x}/x$ from 0 to pi

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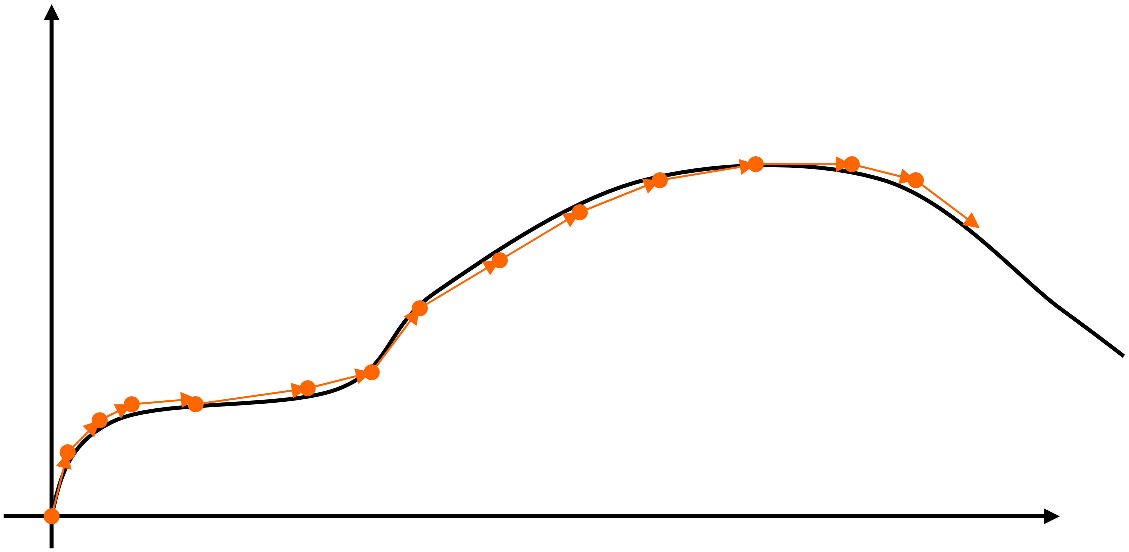
ODE Solvers: Method

- Given a differential equation, the solution can be found by integration:



- Evaluate the derivative at a point and approximate by straight line
- Errors accumulate!
- Variable timestep can decrease the number of iterations

ODE Solvers: MATLAB

- MATLAB contains implementations of common ODE solvers
- Using the correct ODE solver can save you lots of time and give more accurate results
  - » **ode23**
    - Low-order solver. Use when integrating over small intervals or when accuracy is less important than speed
  - » **ode45**
    - High order (Runge-Kutta) solver. High accuracy and reasonable speed. Most commonly used.
  - » **ode15s**
    - Stiff ODE solver (Gear's algorithm), use when the diff eq's have time constants that vary by orders of magnitude

ODE Solvers: Standard Syntax

- To use standard options and variable time step

```
» [t,y]=ode45('myODE',[0,10],[1;0])
```

ODE integrator: 23, 45, 15s

ODE function

Time range

Initial conditions

- Inputs:
  - ODE function name (or anonymous function). This function takes inputs (t,y), and returns dy/dt
  - Time interval: 2-element vector specifying initial and final time
  - Initial conditions: column vector with an initial condition for each ODE. This is the first input to the ODE function
- Outputs:
  - t contains the time points
  - y contains the corresponding values of the integrated fcn.

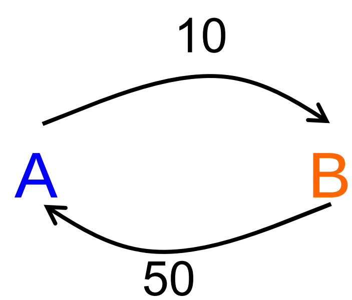
ODE Function

- The ODE function must return the value of the derivative at a given time and function value

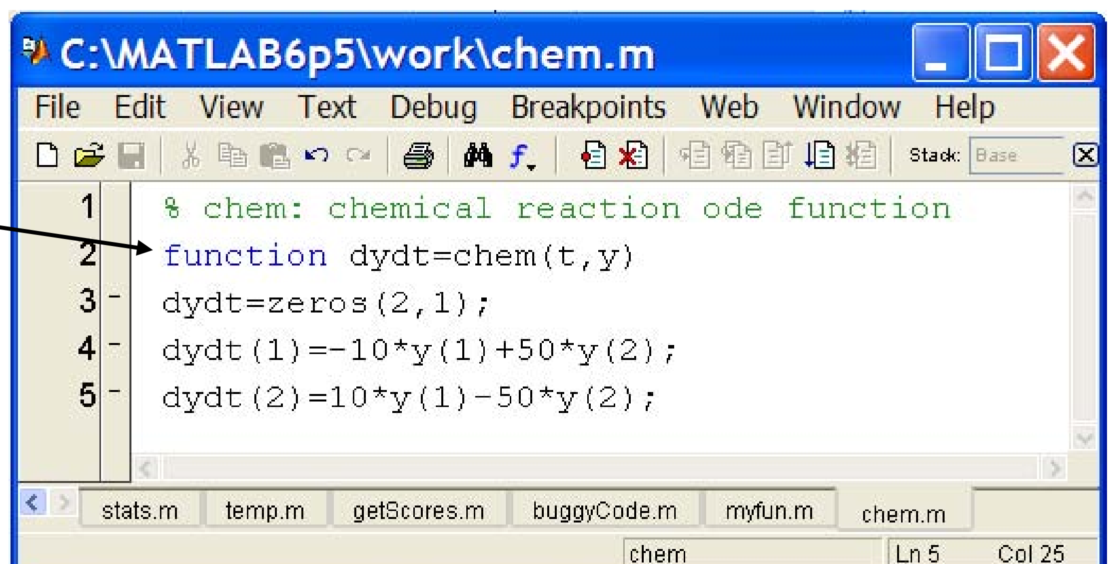
- Example: chemical reaction
  - Two equations

$$\frac{dA}{dt} = -10A + 50B$$

$$\frac{dB}{dt} = 10A - 50B$$



  - ODE file:
    – y has [A;B]
    – dydt has [dA/dt;dB/dt]

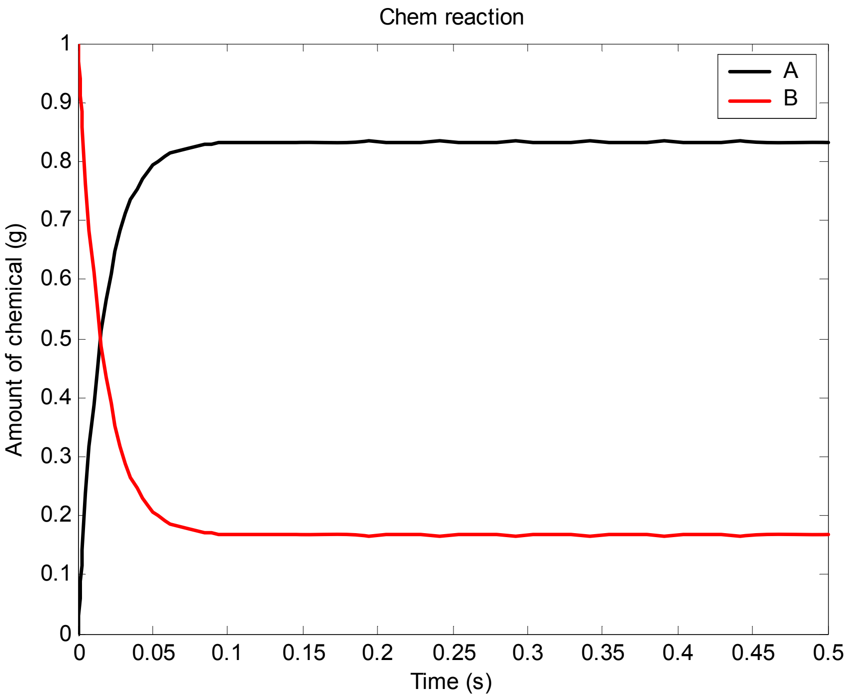


ODE Function: viewing results

- To solve and plot the ODEs on the previous slide:
  - » `[t,y]=ode45('chem',[0 0.5],[0 1]);`
    - ➢ assumes that only chemical B exists initially
  - » `plot(t,y(:,1),'k','LineWidth',1.5);`
  - » `hold on;`
  - » `plot(t,y(:,2),'r','LineWidth',1.5);`
  - » `legend('A','B');`
  - » `xlabel('Time (s)');`
  - » `ylabel('Amount of chemical (g)');`
  - » `title('Chem reaction');`

ODE Function: viewing results

- The code on the previous slide produces this figure



Higher Order Equations

- Must make into a system of first-order equations to use ODE solvers
- Nonlinear is OK!
- Pendulum example:

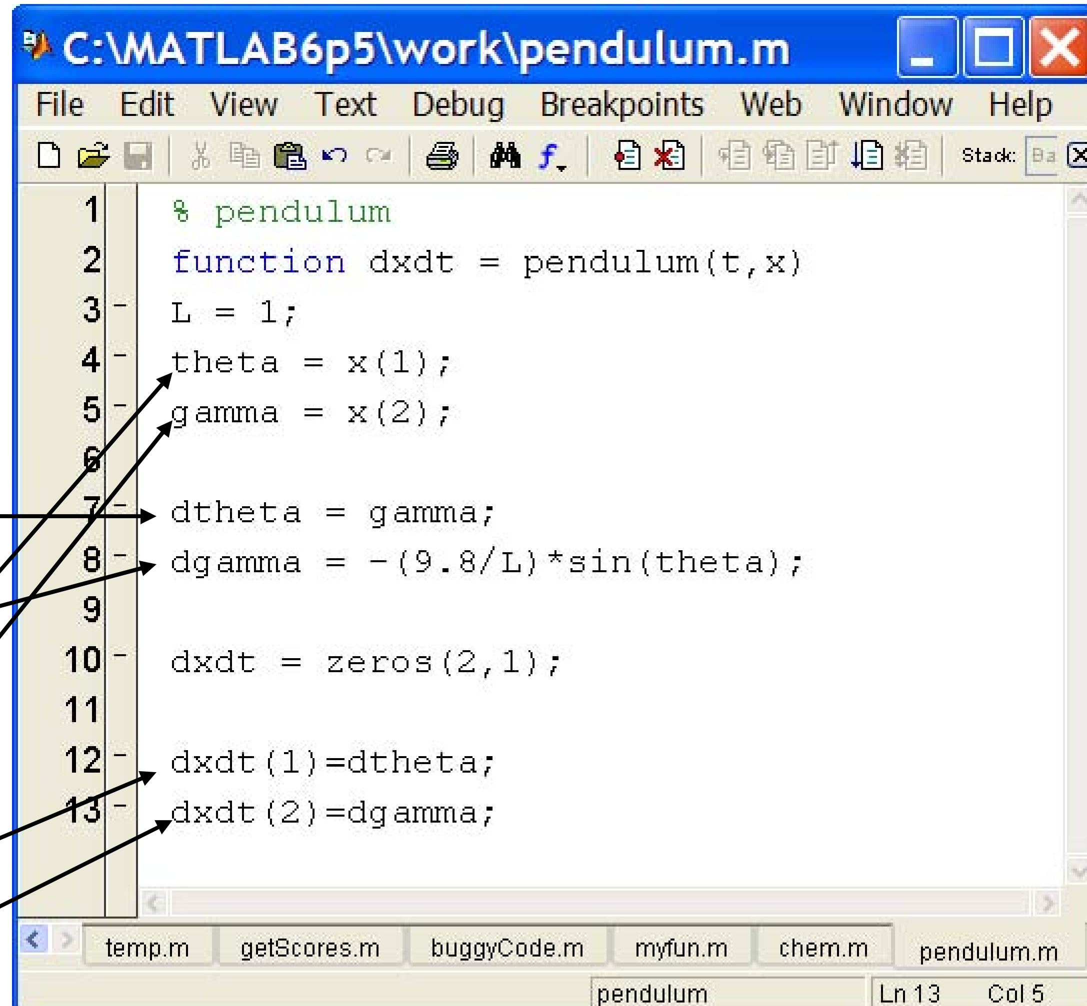
$$\ddot{\theta} + \frac{g}{L}\sin(\theta) = 0$$

$$\ddot{\theta} = -\frac{g}{L}\sin(\theta)$$

$$\text{let} \quad \dot{\theta} = \gamma$$

$$\dot{\gamma} = -\frac{g}{L}\sin(\theta)$$

$$\boxed{\begin{gathered} \vec{x} = \begin{bmatrix} \theta \\ \gamma \end{bmatrix} \\ \frac{d\vec{x}}{dt} = \begin{bmatrix} \dot{\theta} \\ \dot{\gamma} \end{bmatrix} \end{gathered}}$$



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Plotting the Output

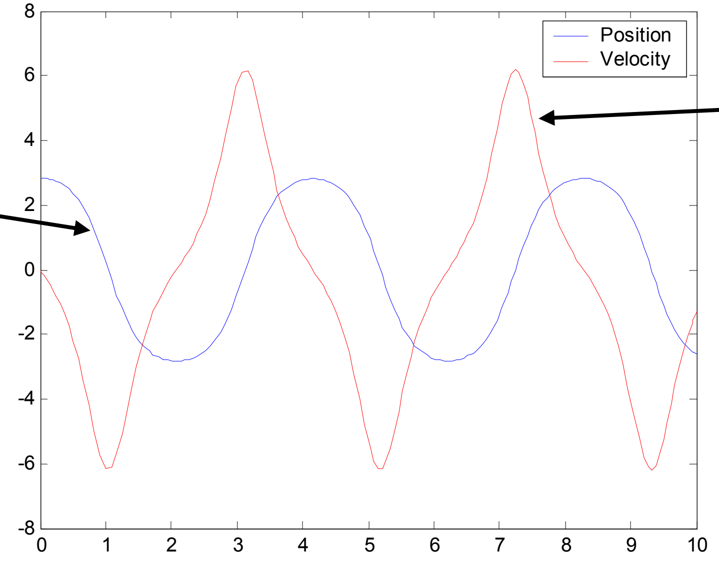
- We can solve for the position and velocity of the pendulum:

```
» [t,x]=ode45('pendulum',[0 10],[0.9*pi 0]);
```

  - ➢ assume pendulum is almost vertical (at top)

```
» plot(t,x(:,1));
» hold on;
» plot(t,x(:,2),'r');
» legend('Position','Velocity');
```

Position in terms of angle (rad)



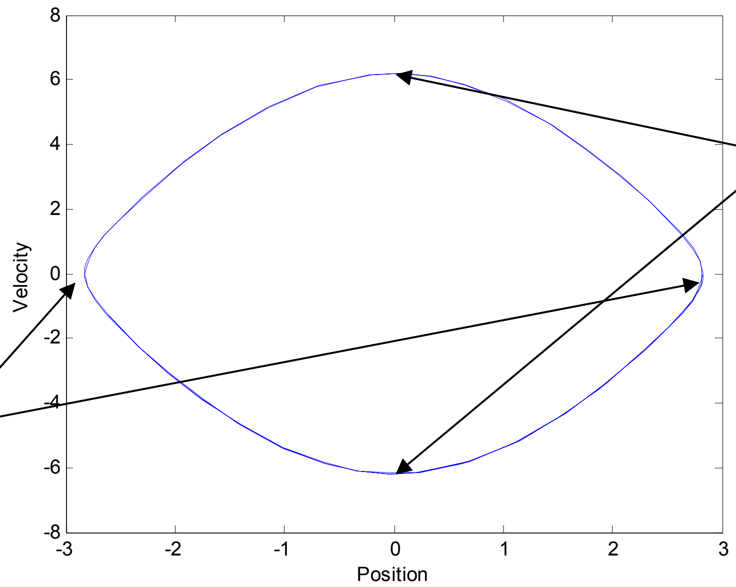
Velocity (m/s)

Plotting the Output

- Or we can plot in the phase plane:

```
» plot(x(:,1),x(:,2));
» xlabel('Position');
» yLabel('Velocity');
```

- The phase plane is just a plot of one variable versus the other:



Velocity is greatest when theta=0

Velocity=0 when theta is the greatest

ODE Solvers: Custom Options

- MATLAB's ODE solvers use a variable timestep
- Sometimes a fixed timestep is desirable
  - » `[t,y]=ode45('chem',[0:0.001:0.5],[0 1]);`
    - ➢ Specify the timestep by giving a vector of times
    - ➢ The function will be evaluated at the specified points
    - ➢ Fixed timestep is usually slower (if timestep is small) and possibly inaccurate (if timestep is too large)

- You can customize the error tolerances using odeset
  - » `options=odeset('RelTol',1e-6,'AbsTol',1e-10);`
  - » `[t,y]=ode45('chem',[0 0.5],[0 1],options);`
    - ➢ This guarantees that the error at each step is less than RelTol times the value at that step, and less than AbsTol
    - ➢ Decreasing error tolerance can considerably slow the solver
    - ➢ See doc odeset for a list of options you can customize

Exercise: ODE

- Use ODE45 to solve this differential equation on the range t=[0 10], with initial condition y(0) = 10: dy/dt=-t*y/10. Plot the result.

Exercise: ODE

- Use ODE45 to solve this differential equation on the range t=[0 10], with initial condition y(0) = 10: dy/dt=-t*y/10. Plot the result.

```
» function dydt=odefun(t,y)
» dydt=-t*y/10;


» [t,y]=ode45(‘odefun’,[0 10],10);
» plot(t,y);
```

End of Lecture 3

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We're almost done!

Issues with ODEs

- Stability and accuracy
    - if step size is too large, solutions might blow up
    - if step size is too small, requires a long time to solve
    - use odeset to control errors
        - decrease error tolerances to get more accurate results
        - increase error tolerances to speed up computation (beware of instability!)
- Main thing to remember about ODEs
    - Pick the most appropriate solver for your problem
    - If ode45 is taking too long, try ode15s