Technical University
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Faculty of Electronics and
Telecommunications

Signals, Circuits and
Systems laboratory

Circuite neliniare pentru prelucrarea semnalelor

# Studiul stabilitatii prin liniarizare locala

# Cuprins

- Studiul stabilitatii prin liniarizare locala
  - Principiul liniarizarii locale
  - Legatura intre ecuatiile de stare liniare si functia de transfer
  - Spatiul starilor; portret de faza
  - Stabiliatatea punctelor de echilibru
  - Stabilitatea punctelor fixe

# Principiul liniarizarii locale

- Functia neliniara de tranzitie a starilor se poate descompune in serie de puteri (Taylor), in jurul unui punct, **x***:

$$\mathbf{f}(\mathbf{x}) = \sum_{n=0}^{\infty} \frac{1}{n!} \cdot \frac{\partial^{\mathbf{n}} \mathbf{f}}{\partial \mathbf{x}^{\mathbf{n}}} \left(\mathbf{x}^*\right) \cdot \left(\mathbf{x} - \mathbf{x}^*\right)^n$$

- Prin detalierea diferentei:

$$\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{x}^*) = \frac{1}{1!} \cdot \frac{\partial \mathbf{f}}{\partial \mathbf{x}} \left(\mathbf{x}^*\right) \cdot \left(\mathbf{x} - \mathbf{x}^*\right) + \frac{1}{2!} \cdot \left(\mathbf{x} - \mathbf{x}^*\right)^T \cdot \frac{\partial^2 \mathbf{f}}{\partial \mathbf{x}^2} \left(\mathbf{x}^*\right) \cdot \left(\mathbf{x} - \mathbf{x}^*\right) + \cdots$$

- Se poate aproxima, intr-o vecinatate restransa a punctului **x***, prin neglijarea puterillor superioare ale argumentului (eventual si a derivatelor de ordin superior):

$$\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{x}^*) \approx \frac{\partial \mathbf{f}}{\partial \mathbf{x}} \left(\mathbf{x}^*\right) \cdot \left(\mathbf{x} - \mathbf{x}^*\right)$$

# Principiul liniarizarii locale (cont.)

- Folosind notatiile:

$$\mathbf{x}-\mathbf{x}^{*}=\mathbf{x}_{l};\quad \mathbf{f}(\mathbf{x})-\mathbf{f}(\mathbf{x}^{*})=\mathbf{f}_{\mathbf{l}}(\mathbf{x}_{l});\quad \frac{\partial\mathbf{f}}{\partial\mathbf{x}}(\mathbf{x}^{*})=\mathbf{J}(\mathbf{x}^{*})=\mathbf{A}$$

- Obtinem sistemul liniarizat: $\mathbf{f}_{\mathbf{l}}(\mathbf{x}_{l})\approx\mathbf{A}\cdot\mathbf{x}_{l}$

- Unde Jacobianul sistemului este:

$$\mathbf{J}(\mathbf{x})=\partial\mathbf{f}/\partial\mathbf{x}=\begin{pmatrix}\partial f_1/\partial x_1 & \partial f_1/\partial x_2 & \cdots & \partial f_1/\partial x_N\\ \partial f_2/\partial x_1 & \partial f_2/\partial x_2 & \cdots & \partial f_2/\partial x_N\\ \vdots & \vdots & \ddots & \vdots\\ \partial f_N/\partial x_1 & \partial f_N/\partial x_2 & \cdots & \partial f_N/\partial x_N\end{pmatrix}$$

# Principiul liniarizarii locale (cont.)

- Pentru sisteme analogice, se obtin ecuatiile de stare liniarizate:

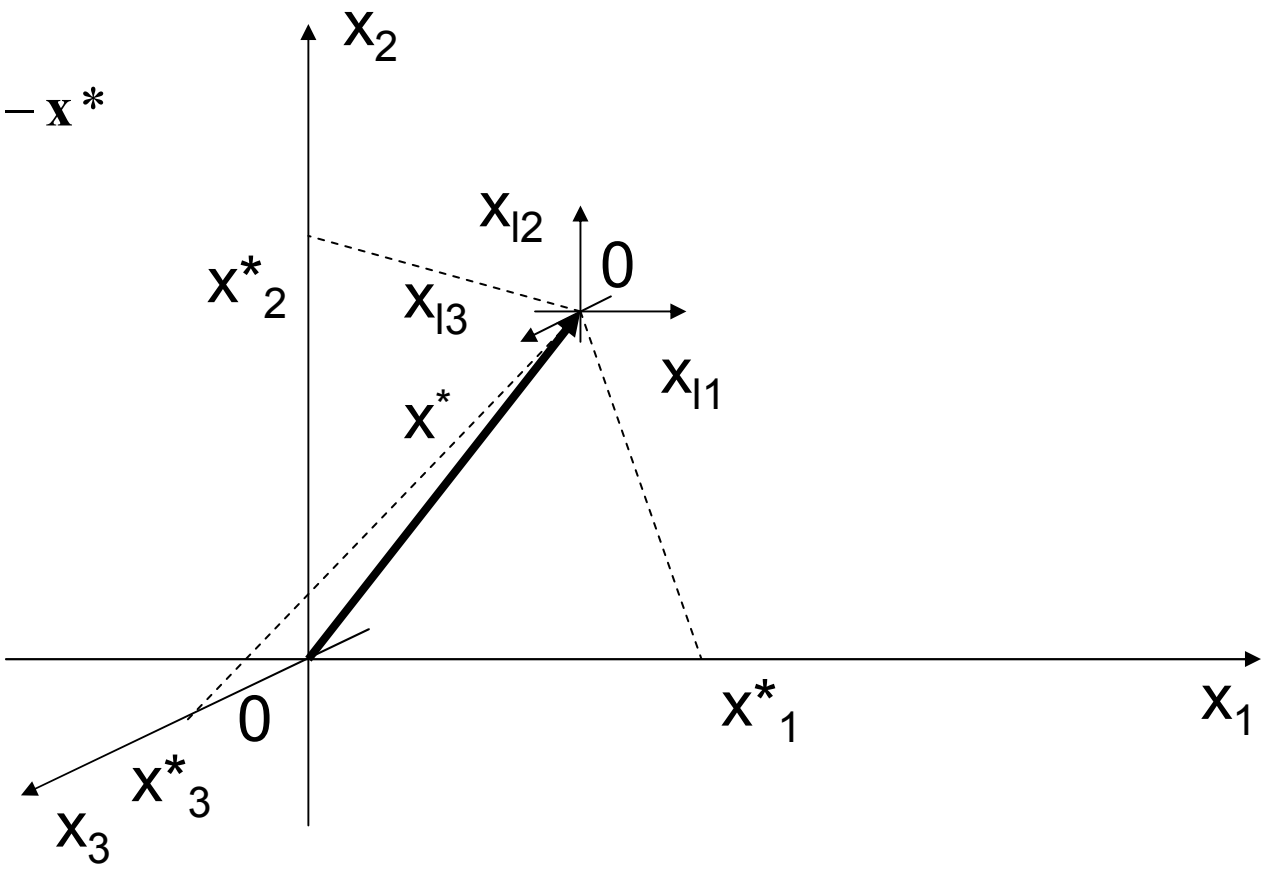
$$\left.\begin{array}{l} \mathbf{x}'(t) = \mathbf{f}\big(\mathbf{x}(t)\big) \\ \mathbf{f}_{\mathbf{l}}\big(\mathbf{x}_{\mathbf{l}}\big) \approx \mathbf{A} \cdot \mathbf{x}_{\mathbf{l}} \end{array}\right\} \Rightarrow \quad \mathbf{x}_l{}' = \mathbf{A} \cdot \mathbf{x}_l$$

- Pentru sisteme discrete:

$$\left.\begin{array}{l} \mathbf{x}[k+1] = \mathbf{f}(\mathbf{x}[k]) \\ \mathbf{f}_{\mathbf{l}}\big(\mathbf{x}_l\big) \approx \mathbf{A} \cdot \mathbf{x}_l \end{array}\right\} \Rightarrow \quad \mathbf{x}_l[k+1] = \mathbf{A} \cdot \mathbf{x}_l[k]$$

# Interpretare grafica a liniarizarii locale

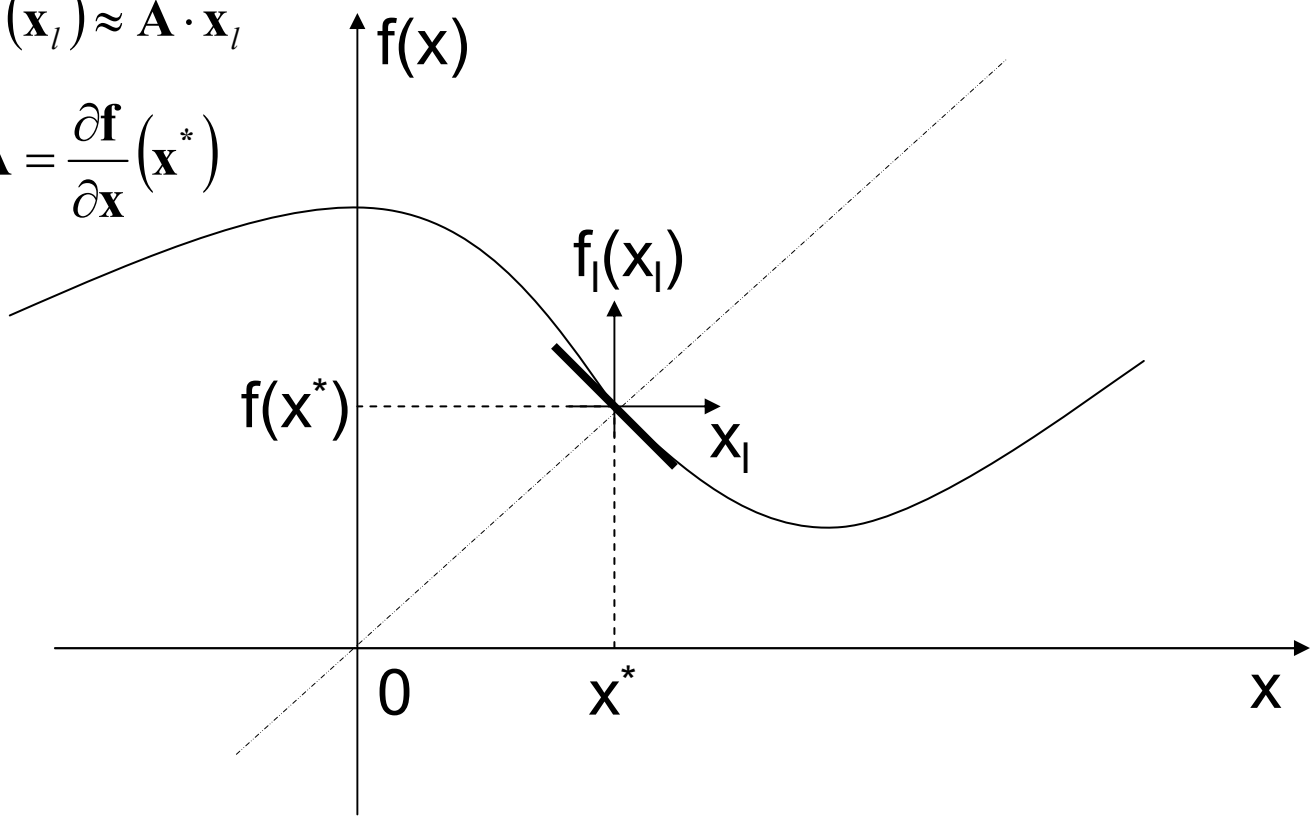
$$\mathbf{x}_l = \mathbf{x} - \mathbf{x}^*$$



# Interpretare grafica a liniarizarii locale

$$\mathbf{f}_l(\mathbf{x}_l) \approx \mathbf{A} \cdot \mathbf{x}_l$$

$$\mathbf{A} = \frac{\partial \mathbf{f}}{\partial \mathbf{x}}(\mathbf{x}^*)$$



# Legatura ecuatii de stare liniare <-> functie de transfer

- Sistemul ne-autonom liniar analogic: $\begin{cases} \mathbf{x}'(t) = \mathbf{A} \cdot \mathbf{x}(t) + \mathbf{b} \cdot e(t) \\ y(t) = \mathbf{c}^T \cdot \mathbf{x}(t) + d \cdot e(t) \end{cases}$
- Este caracterizat in transformata Laplace de ecuatiile algebrice:

$$\begin{cases} s\mathbf{X}(s) = \mathbf{A} \cdot \mathbf{X}(s) + \mathbf{b} \cdot E(s) \\ Y(s) = \mathbf{c}^T \cdot \mathbf{X}(s) + d \cdot E(s) \end{cases}$$

- Din ecuatiile de stare se determina transformata Laplace a vectorului de stare: $(s\mathbf{I} - \mathbf{A}) \cdot \mathbf{X}(s) = \mathbf{b} \cdot E(s)$

$$\mathbf{X}(s) = (s\mathbf{I} - \mathbf{A})^{-1} \cdot \mathbf{b} \cdot E(s)$$

- Inlocuind in ecuatia de iesire: $Y(s) = \mathbf{c}^T \cdot (s\mathbf{I} - \mathbf{A})^{-1} \cdot \mathbf{b} \cdot E(s) + d \cdot E(s)$
- Se obtine functia de transfer: $H(s) = \dfrac{Y(s)}{E(s)} = \mathbf{c}^T \cdot (s\mathbf{I} - \mathbf{A})^{-1} \cdot \mathbf{b} + d$

# Legatura valori proprii <-> poli

- Prin calcul algebric: $H(s) = \mathbf{c}^T \cdot (s\mathbf{I} - \mathbf{A})^{-1} \cdot \mathbf{b} + d$

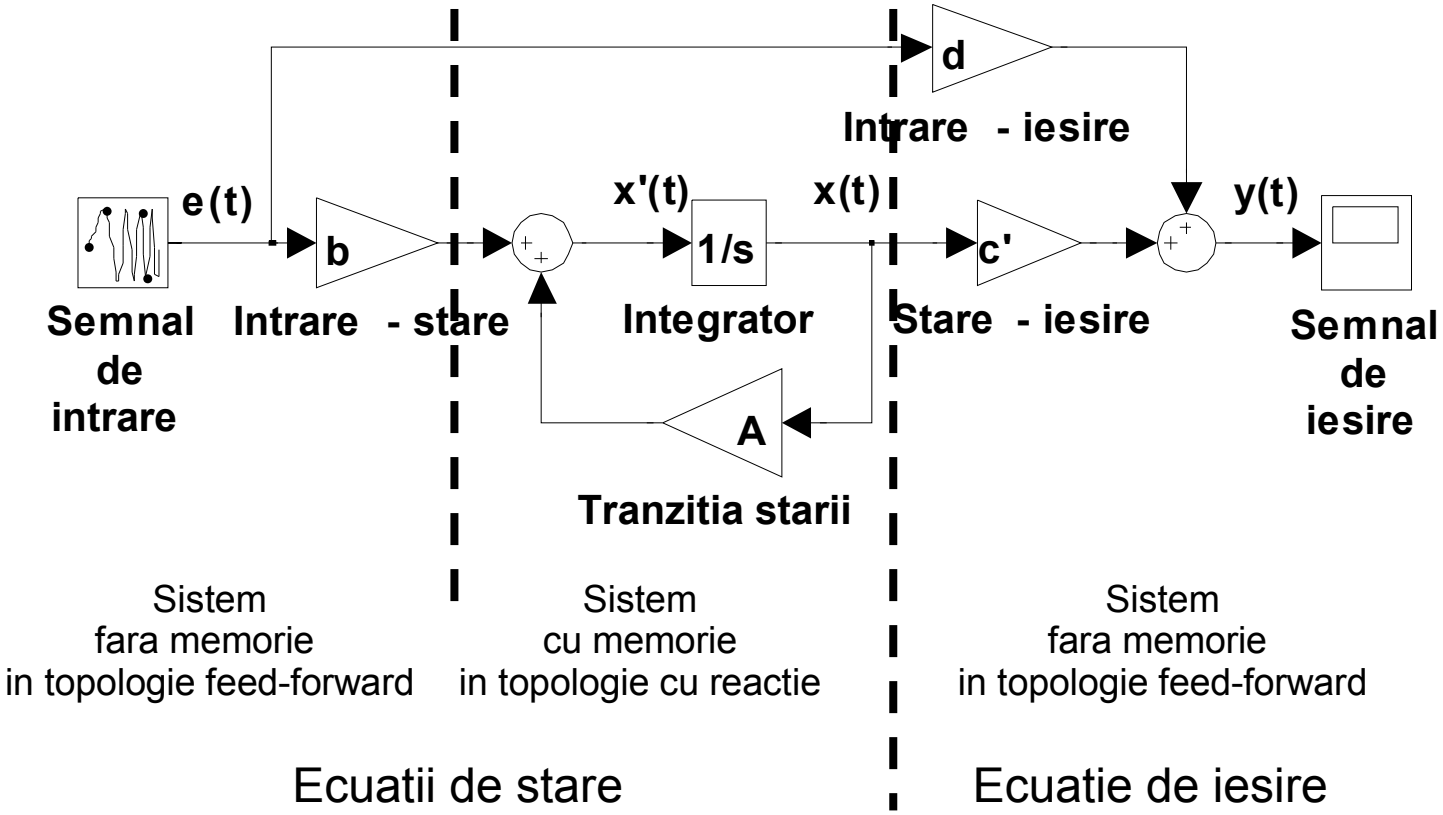
$$(s\mathbf{I} - \mathbf{A})^{-1} = \frac{Adj(s\mathbf{I} - \mathbf{A})^T}{\det(s\mathbf{I} - \mathbf{A})} \qquad H(s) = \frac{\mathbf{c}^T \cdot Adj(s\mathbf{I} - \mathbf{A})^T \cdot \mathbf{b} + d \cdot \det(s\mathbf{I} - \mathbf{A})}{\det(s\mathbf{I} - \mathbf{A})} = \frac{P_M(s)}{Q_N(s)}$$

- **Polii** functiei de transfer: $Q_N(s) = 0 \quad \Leftrightarrow \quad \det(s\mathbf{I} - \mathbf{A}) = 0$

- Sunt **valorile proprii** ale matricei de tranzitie a starilor:

$$\mathbf{A} \cdot \mathbf{v} = s \cdot \mathbf{v} \quad \Leftrightarrow \quad (\mathbf{A} - s\mathbf{I}) \cdot \mathbf{v} = 0$$

# Interpretarea la nivel de schema bloc



# Legatura ecuatii de stare liniare <-> functie de transfer

- Sistemul ne-autonom liniar discret: $\begin{cases} \mathbf{x}[k+1] = \mathbf{A} \cdot \mathbf{x}[k] + \mathbf{b} \cdot e[k] \\ y[k] = \mathbf{c}^T \cdot \mathbf{x}[k] + d \cdot e[k] \end{cases}$
- Este caracterizat in transformata Z de ecuatiile algebrice:

$$\begin{cases} z\mathbf{X}(z) = \mathbf{A} \cdot \mathbf{X}(z) + \mathbf{b} \cdot E(z) \\ Y(z) = \mathbf{c}^T \cdot \mathbf{X}(z) + d \cdot E(z) \end{cases}$$

- Din ecuatiile de stare se determina transformata Z a vectorului de stare: $(z\mathbf{I} - \mathbf{A}) \cdot \mathbf{X}(z) = \mathbf{b} \cdot E(z)$

$$\mathbf{X}(z) = (z\mathbf{I} - \mathbf{A})^{-1} \cdot \mathbf{b} \cdot E(z)$$

- Inlocuind in ecuatia de iesire: $Y(z) = \mathbf{c}^T \cdot (z\mathbf{I} - \mathbf{A})^{-1} \cdot \mathbf{b} \cdot E(z) + d \cdot E(z)$
- Se obtine functia de transfer: $H(z) = \dfrac{Y(z)}{E(z)} = \mathbf{c}^T \cdot (z\mathbf{I} - \mathbf{A})^{-1} \cdot \mathbf{b} + d$

# Legatura valori proprii <-> poli

- Prin calcul algebric: $H(z) = \mathbf{c}^T \cdot (z\mathbf{I} - \mathbf{A})^{-1} \cdot \mathbf{b} + d$

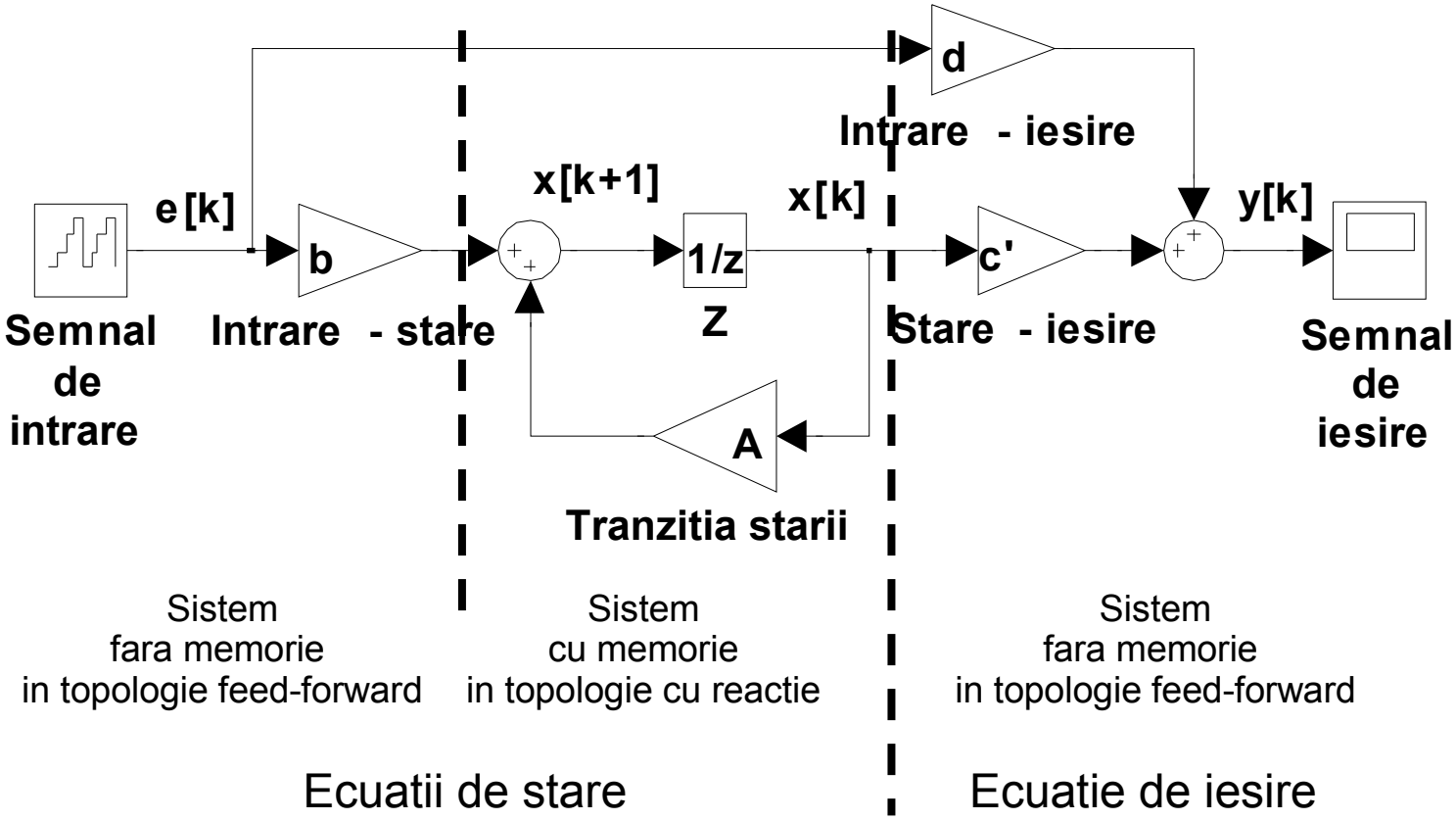
$$(z\mathbf{I} - \mathbf{A})^{-1} = \frac{Adj(z\mathbf{I} - \mathbf{A})^T}{\det(z\mathbf{I} - \mathbf{A})} \quad H(z) = \frac{\mathbf{c}^T \cdot Adj(z\mathbf{I} - \mathbf{A})^T \cdot \mathbf{b} + d \cdot \det(z\mathbf{I} - \mathbf{A})}{\det(z\mathbf{I} - \mathbf{A})} = \frac{P_M(z)}{Q_N(z)}$$

- **Polii** functiei de transfer: $Q_N(z) = 0 \quad \Leftrightarrow \quad \det(z\mathbf{I} - \mathbf{A}) = 0$

- Sunt **valorile proprii** ale matricei de tranzitie a starilor:

$$\mathbf{A} \cdot \mathbf{v} = z \cdot \mathbf{v} \quad \Leftrightarrow \quad (\mathbf{A} - z\mathbf{I}) \cdot \mathbf{v} = 0$$

# Interpretarea la nivel de schema bloc



# Spatiul starilor; portret de faza

- **Definitie:** spatiul vectorial geometric, N – dimensional al vectorilor de stare **x**, se numeste **spatiul starilor**.
- **Definitie:** reprezentarea grafica a evolutiei temporale, a vectorului de stare, N – dimensional, **x**, se numeste **portret de faza.**
- Forma canonica **Jordan**:

$$\mathbf{A} = \mathbf{U} \cdot \mathbf{J} \cdot \mathbf{U}^{-1}$$

$$\mathbf{J} = \mathbf{U}^{-1} \cdot \mathbf{A} \cdot \mathbf{U}$$

$$\mathbf{J} = \begin{pmatrix} \mathbf{A}_1 & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_2 & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{A}_{N/2} \end{pmatrix} \qquad \mathbf{A}_n \in \mathbf{M}_{2x2}$$

- Permite partitionarea unui sistem de ordin superior in N/2 subsisteme de ordinul 2

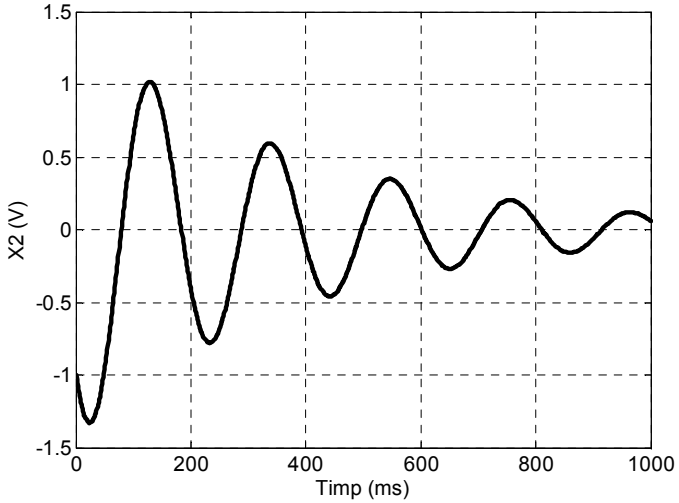
$$\mathbf{x} = \mathbf{U} \cdot \mathbf{x}_r$$

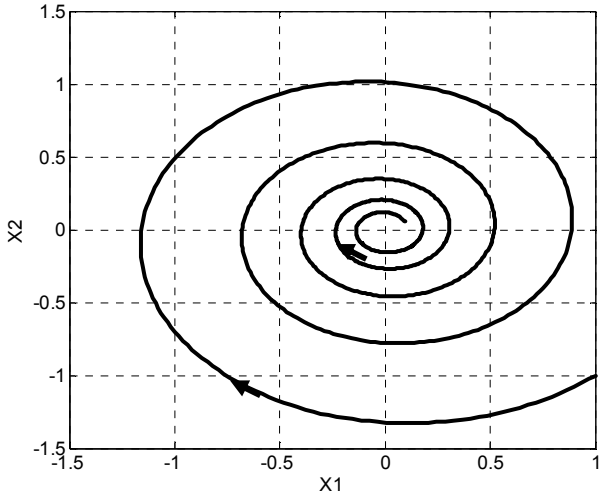
$$\begin{cases} \mathbf{x}_r'(t) = \mathbf{U}^{-1} \cdot \mathbf{A} \cdot \mathbf{U} \cdot \mathbf{x}(t) + \mathbf{U}^{-1} \cdot \mathbf{b} \cdot e(t) \\ y(t) = \mathbf{c}^T \cdot \mathbf{U}^{-1} \cdot \mathbf{x}(t) + d \cdot e(t) \end{cases}$$

$$\begin{cases} \mathbf{x}'(t) = \mathbf{A} \cdot \mathbf{x}(t) + \mathbf{b} \cdot e(t) \\ y(t) = \mathbf{c}^T \cdot \mathbf{x}(t) + d \cdot e(t) \end{cases}$$

$$\begin{cases} \mathbf{x}_r'(t) = \mathbf{J} \cdot \mathbf{x}_r(t) + \mathbf{b}_r \cdot e(t) \\ y(t) = \mathbf{c}_r^{\ T} \cdot \mathbf{x}_r(t) + d \cdot e(t) \end{cases}$$

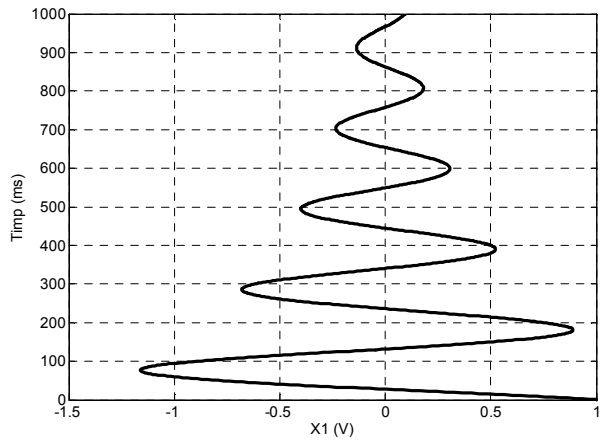
# Portret de faza pentru un bloc 2x2



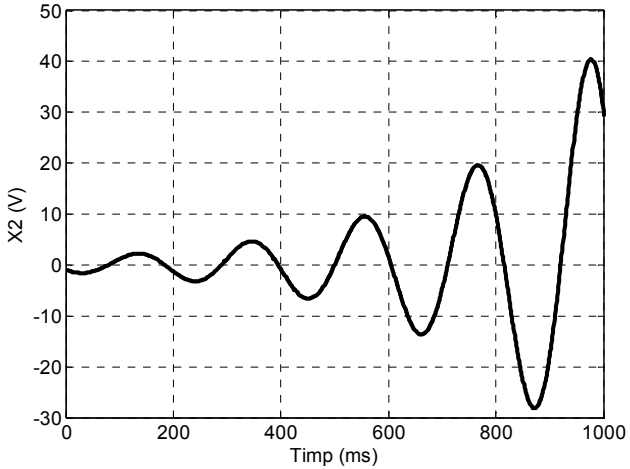



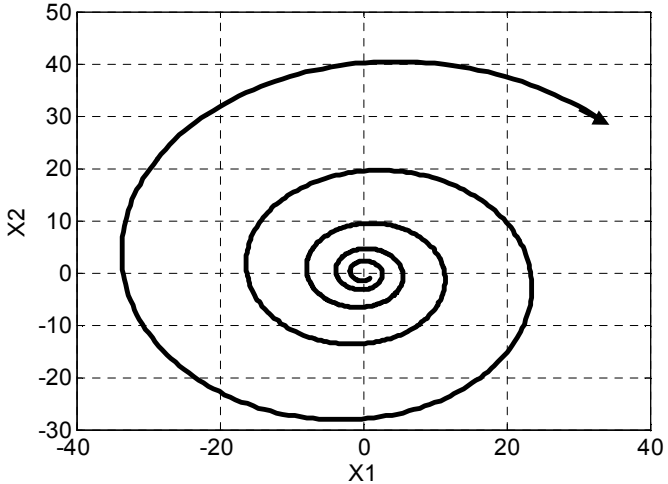

Focar stabil

$$s_{1,2} \in C \quad \mathrm{Re}\left(s_{1,2}\right) < 0$$



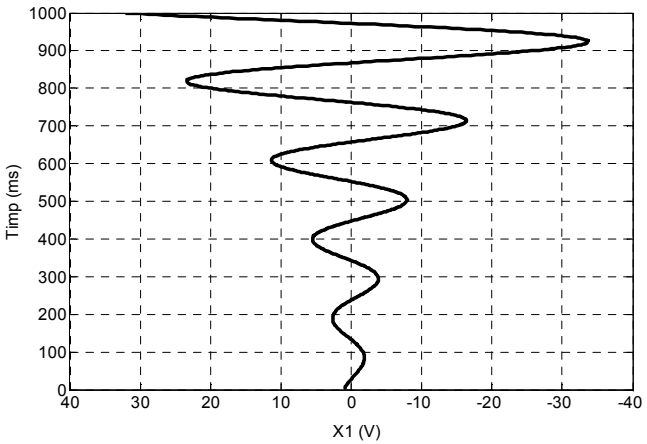
# Portret de faza pentru un bloc 2x2



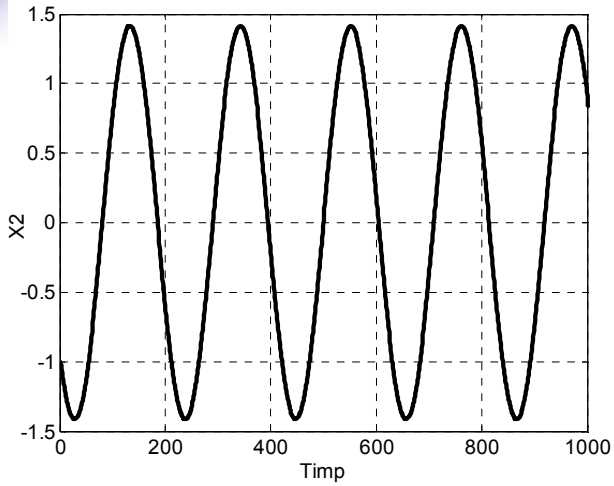



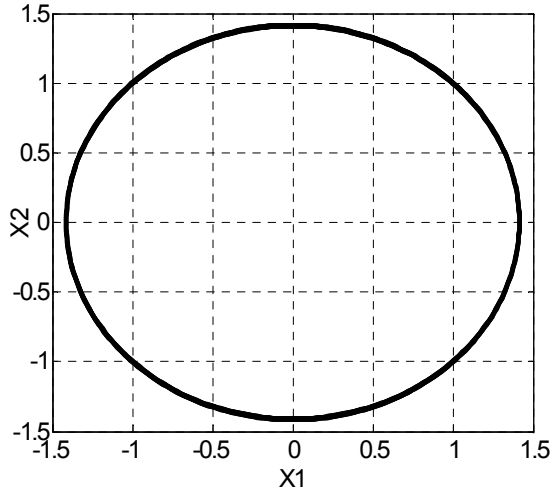

Focar instabil

$$s_{1,2} \in C \quad \mathrm{Re}(s_{1,2}) > 0$$



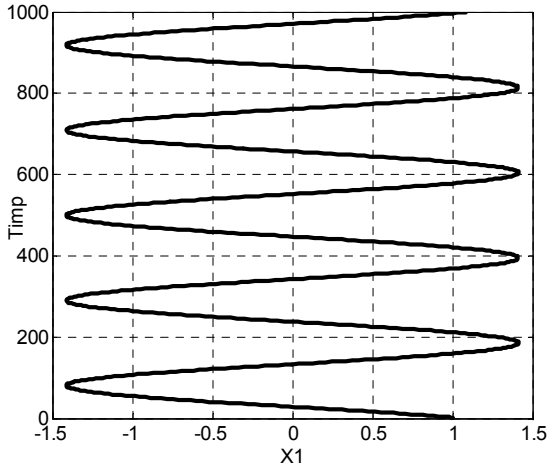
# Portret de faza pentru un bloc 2x2



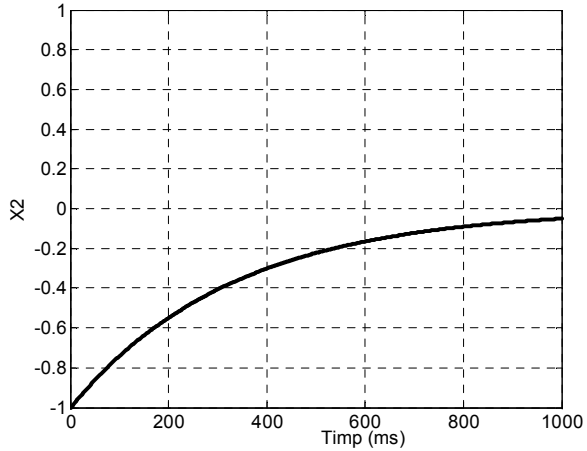



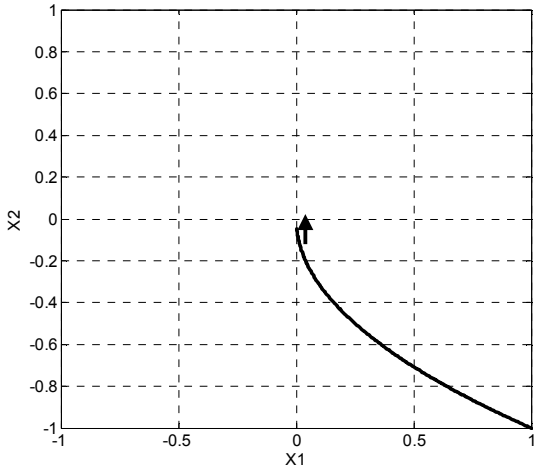

Centru

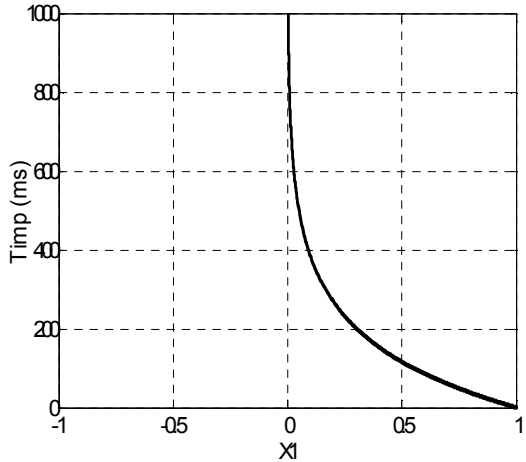
$$s_{1,2} \in C \quad \mathrm{Re}\left(s_{1,2}\right) = 0$$



# Portret de faza pentru un bloc 2x2



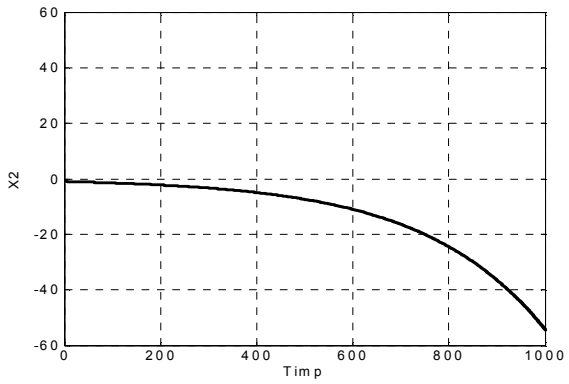



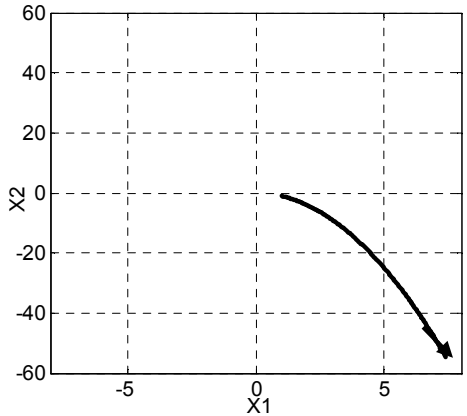



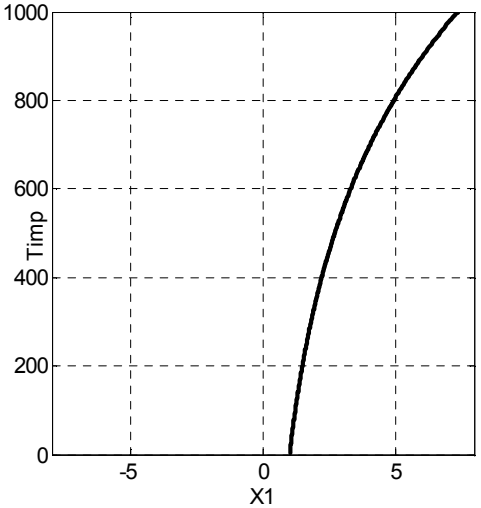

Nod stabil

$$s_{1,2} \in \Re \quad s_{1,2} < 0$$

# Portret de faza pentru un bloc 2x2



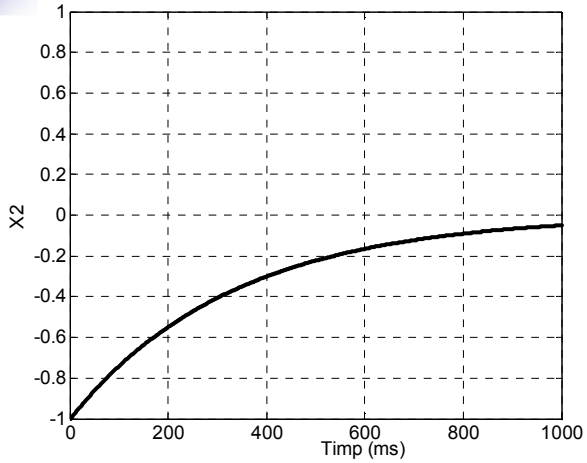



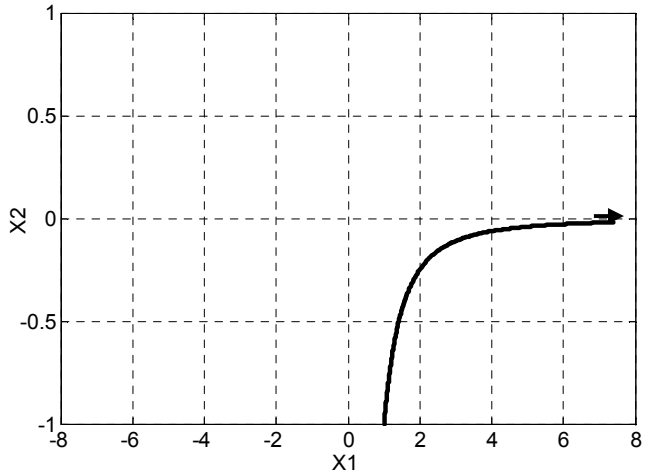




Nod instabil

$$s_{1,2} \in \Re \quad s_{1,2} > 0$$

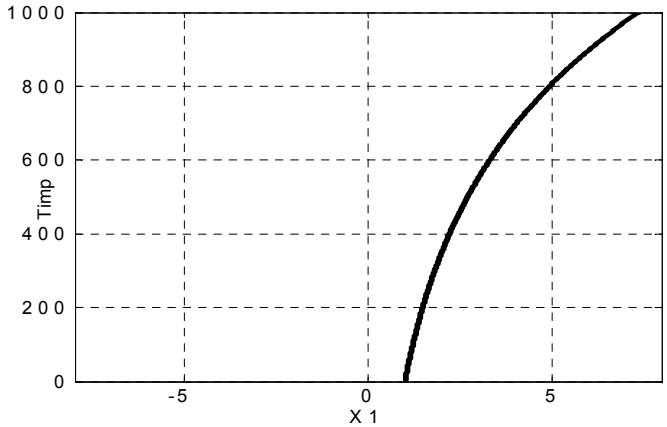
# Portret de faza pentru un bloc 2x2




Punct sea

$$s_{1,2} \in \Re \quad s_1 > 0 \quad s_2 < 0$$



# Stabilitatea punctelor de echilibru

- Pentru sisteme analogice liniarizate:

$$\mathbf{x}' = \mathbf{A} \cdot \mathbf{x}$$

- Se calculeaza valorile proprii ale matricii de tranzitie a starilor:

$$\det\left(s \cdot \mathbf{I_N} - \mathbf{A}\right) = 0 \Rightarrow s \in \left\{s_n\right\}_{n=1,\ldots,N} \subset \mathbf{C}$$

- Reprezentand polii functiei de transfer a circuitului.

**Teoremă**:

- Dacă sistemul liniarizat, în jurul punctului de echilibru, **x*,** este strict stabil, punctul de echilibru, **x*,** este atractor.
- Dacă sistemul liniarizat în jurul punctului de echilibru, **x*,** este strict instabil, punctul de echilibru, **x*,** este repulsor.
- Cazul limitei de stabilitate este indecidabil.

# Metodologia de analiza a stabilitatii punctelor de echilibru

1. Determinarea punctelor de echilibru ale sistemului neliniar analogic:

$$\mathbf{x}'(t) = \mathbf{f}(\mathbf{x}(t)) \qquad \mathbf{f}(\mathbf{x}) = 0$$

2. Alegerea unuia dintre punctele de echilibru, **x***.
3. Calculul Jacobianului: J(**x***) = **A**

$$\frac{\partial \mathbf{f}}{\partial \mathbf{x}}(\mathbf{x}^*) = \mathbf{J}(\mathbf{x}^*) = \mathbf{A}$$

4. Calculul valorilor proprii ale matricei de tranzitie a starilor sistemului liniarizat:

$$\det(s\mathbf{I_N} - \mathbf{A}) = 0 \quad \Rightarrow \quad s = s_n\big|_{n=1,..,N}$$

5. Analiza stabilitatii punctului de echilibru prin determinarea stabilitatii sistemului liniarizat:

$$\mathrm{Re}(s_n) < 0 \quad \forall n = 1,..,N$$

# Stabilitatea punctelor fixe

- Pentru sisteme discrete liniarizate: $\mathbf{x}[k+1] = \mathbf{A} \cdot \mathbf{x}[k]$
- Se pot determina valorile proprii ale matricei de tranzitie a starilor:

$$\det(z \cdot \mathbf{I_N} - \mathbf{A}) = 0 \Rightarrow z \in \{z_n\}_{n=1,\ldots,N} \subset \mathbf{C}$$

- Reprezentand polii functiei de transfer a circuitului.

**Teoremă**

- Dacă sistemul discret, liniarizat în jurul punctului fix, **x*** este strict stabil, punctul fix **x*** este atractor.
- Dacă sistemul discret liniarizat în jurul punctului fix **x*** este strict instabil, punctul fix **x*** este repulsor.
- Cazul limitei de stabilitate este indecidabil.

# Metodologia de analiza a stabilitatii punctelor fixe

1. Determinarea punctelor fixe ale sistemului neliniar discret:

$$\mathbf{x}[k+1] = \mathbf{f}(\mathbf{x}[k]) \qquad \mathbf{f}(\mathbf{x}) = \mathbf{x}$$

2. Alegerea unuia dintre punctele fixe, **x***.
3. Calculul Jacobianului: J(**x***) = **A** $\quad \frac{\partial \mathbf{f}}{\partial \mathbf{x}}(\mathbf{x}^*) = \mathbf{J}(\mathbf{x}^*) = \mathbf{A}$
4. Calculul valorilor proprii ale matricei de tranzitie a starilor sistemului liniarizat:

$$\det(z\mathbf{I_N} - \mathbf{A}) = 0 \quad \Rightarrow \quad z = z_n\big|_{n=1,..,N}$$

5. Analiza stabilitatii punctului de echilibru prin determinarea stabilitatii sistemului liniarizat:

$$|z_n| < 1 \quad \forall n = 1,..,N$$

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