Phase Influence on Mode Competition in Turing Pattern Formation

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Abstract

Within linear theory, Turing patterns in CNN's can be viewed as the consequence of a competition between unstable spatial modes. The aim of this communication is to show that the final pattern might depend on the relative position (phase) of the competing modes – a result that cannot be explained using the mode decoupling linear theory.

INTRODUCTION

The two-grid coupled Cellular Neural Networks (CNN's) architecture [1-11] has been shown to be capable to produce Turing patterns on the basis of a mechanism similar to that proposed by Turing [14]. Composed of identical cells identically coupled by means of two homogeneous resistive grids, such CNN's exhibit an **unstable** homogeneous equilibrium point, which corresponds to a **stable** one for an isolated cell. The pattern is one of the stable equilibrium points towards which the network emerges. The linearized equations governing the dynamics of the array have the form:

$$\begin{aligned} \frac{du_{ij}(t)}{dt} &= \gamma(f_u u_{ij} + f_v v_{ij}) + D_u \nabla^2 u_{ij} \\ \frac{dv_{ij}(t)}{dt} &= \gamma(g_u u_{ij} + g_v v_{ij}) + D_v \nabla^2 v_{ij} \end{aligned} \qquad (1)$$

where f_u, f_v, g_u, g_v refer to the linearized two-port resistive characteristics (elements of the Jacobian matrix of *f(u,v)* and *g(u,v)* of the nonlinear equations), D_u and D_v are the diffusion coefficients and ∇^2 is the discrete Laplacian. The Turing conditions [3-8], that have been shown to be only necessary for discrete arrays, are:

$$\begin{aligned} & f_u + g_v < 0 \\ & f_u g_v - f_v g_u > 0 \\ & D_v f_u + D_u g_v > 0 \\ & (D_v f_u - D_u g_v)^2 + 4 D_u D_v f_v g_u > 0 \end{aligned} \qquad (2)$$

Within linear theory, Turing Patterns in CNN's are dependent on the following aspects:

a – fulfillment of Turing conditions,

b – dispersion curve,

c – initial conditions,

d – biasing sources signal [3]

Beside, the shape of the nonlinear characteristic of the cell resistor influences the pattern as well but this is an aspect that cannot be consider within the above theory. However, it has been shown that the results of the linear theory fit well with the simulations mainly for 1D arrays. In such cases, the final pattern can usually be predicted taking into consideration the above aspects, which means that the nonlinearity ($f_u(u)$ in most cases – as shown in Fig.1.) plays mainly the role of limiting the growing process of the unstable spatial modes.

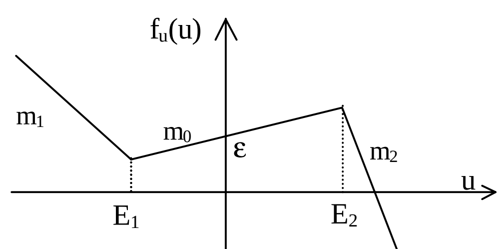

Fig. 1: A typical cell non-linear characteristic

Using the decoupling technique, i.e., the change of variable

$$\begin{aligned} u_i(t) &= \sum_{m=0}^{M-1} \Phi_M(m,i)\hat{u}_m(t) \\ v_i(t) &= \sum_{m=0}^{M-1} \Phi_M(m,i)\hat{v}_m(t) \end{aligned} \tag{3}$$

i=0,1,…,M-1, where the functions $\Phi_M(m,i)$ are dependent on the boundary conditions as shown in [5]

In terms of the new variables, the dynamics of the CNN is described by the following set of pairs of decoupled linearized equations

$$\begin{bmatrix} \dot{\hat{u}}_m(t) \\ \dot{\hat{v}}_m(t) \end{bmatrix} = \left(\gamma \begin{bmatrix} f_u & f_v \\ g_u & g_v \end{bmatrix} - k_m^2 \begin{bmatrix} D_u & 0 \\ 0 & D_v \end{bmatrix} \right) \begin{bmatrix} \hat{u}_m(t) \\ \hat{v}_m(t) \end{bmatrix} + \gamma \begin{bmatrix} \hat{\varepsilon}_m \\ 0 \end{bmatrix} \tag{4}$$

The general form of the transient expressed in terms of the decoupled variables [9]

$$\begin{cases} \hat{u}_m(t) = a_m(\hat{\varepsilon}_m, \hat{u}_m(0), \hat{v}_m(0)) e^{\lambda_{m_1} t} + b_m(\hat{\varepsilon}_m, \hat{u}_m(0), \hat{v}_m(0)) e^{\lambda_{m_2} t} + f_1(\hat{\varepsilon}_m) \\ \hat{v}_m(t) = c_m(\hat{\varepsilon}_m, \hat{u}_m(0), \hat{v}_m(0)) e^{\lambda_{m_1} t} + d_m(\hat{\varepsilon}_m, \hat{u}_m(0), \hat{v}_m(0)) e^{\lambda_{m_2} t} + f_2(\hat{\varepsilon}_m) \end{cases} \tag{5}$$

where λ_{m1} and λ_{m2} are the roots of the characteristic equations

$$\begin{aligned} &\lambda_m^2 + \lambda_m[k_m^2(D_u + D_v) - \gamma(f_u + g_v)] + D_u D_v k_m^4 \\ &- \gamma(D_v f_u + D_u g_v)k_m^2 + (f_u g_v - f_v g_u) = 0 \end{aligned} \tag{6}$$

and f_1 and f_2 are [9]

$$\begin{cases} f_1(\hat{\varepsilon}_m) = \dfrac{-\gamma(\gamma g_v - k_m^2 D_v)}{(\gamma f_u - k_m^2 D_u)(\gamma g_v - k_m^2 D_v) - \gamma^2 f_v g_u} \hat{\varepsilon}_m \\ f_2(\hat{\varepsilon}_m) = \dfrac{\gamma^2 g_u}{(\gamma f_u - k_m^2 D_u)(\gamma g_v - k_m^2 D_v) - \gamma^2 f_v g_u} \hat{\varepsilon}_m \end{cases} \tag{7}$$

The time domain solution of the 1-D linearized CNN equations is thus

$$\begin{aligned} u_i(t) &= \sum_{m=0}^{M-1} (a_m(\hat{\varepsilon}_m, \hat{u}_m(0), \hat{v}_m(0)) e^{\lambda_{m_1} t} + b_m(\hat{\varepsilon}_m, \hat{u}_m(0), \hat{v}_m(0)) e^{\lambda_{m_2} t} + f_1(\hat{\varepsilon}_m)) \Phi_M(m,i) \\ v_i(t) &= \sum_{m=0}^{M-1} (c_m(\hat{\varepsilon}_m, \hat{u}_m(0), \hat{v}_m(0)) e^{\lambda_{m_1} t} + d_m(\hat{\varepsilon}_m, \hat{u}_m(0), \hat{v}_m(0)) e^{\lambda_{m_2} t} + f_2(\hat{\varepsilon}_m)) \Phi_M(m,i) \end{aligned} \tag{8}$$

For ring boundary conditions, the orthogonal basis of functions is: $\Phi_M(m,i) = e^{j\frac{2\pi}{M}mi}$ and the corresponding eigenvalues, $k_m^2 = 4\sin^2\frac{m\pi}{M}$.

The dispersion curve represents the real part of the temporal eigenvalues versus the spatial eigenvalues.

A typical dispersion curve is represented in the figure below:

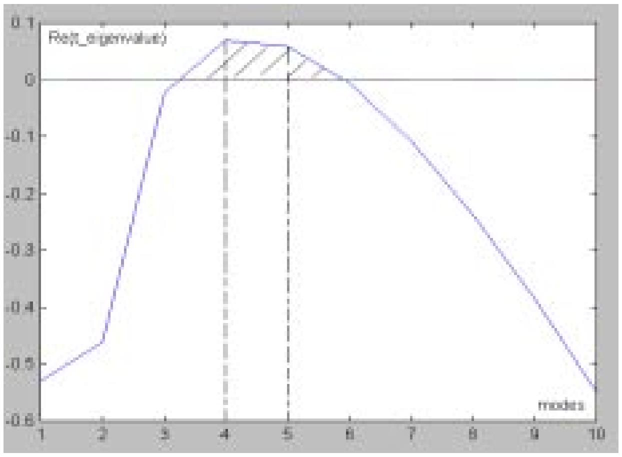

Fig. 2: A typical dispersion curve

For the situation in the figure, there are two “active” modes: 4 and 5. We say that they are “inside” the dispersion curve, i.e. they have eigenvalues with positive real parts.
When using non-homogeneous spatial bias current sources, the “spatial signal” made by bias current sources influence the solution. We stress the fact that the functioning point is before entering the non-linear part of each cell’s characteristic.

PHASE INFLUENCE ON MODE COMPETITION

For initial conditions consisting of two pure spatial modes the technique of decoupling the differential equations predicts a race between the spatial modes which depends on their weight in the initial conditions and on the magnitude of the (positive) real parts of the corresponding temporal modes. The relative position of the two spatial modes is irrelevant within the linear theory. This statement is true as far as the amplification conditions for one mode are much more favourable than for the other one (in terms of amplitude ratio and eigenvalue real parts) [10-13].
However, when the competition is “tight”, it has been found that the relative position (phase) of the two competing modes can influence the final pattern.
The simulations have been done with the following parameters:

1D size	fu	fv	gu	gv	gamma	du	dv
50	0.4	1	-0.25	-0.5	1	1	10

Parameters deduced from the dipersion curve:

peak	k1	k2	Dvcrit	kcrit	m=1 (IN)	m=2 (IN)	m=3 (IN)	m=4 (IN)
0.14864@ 0.09325	0.01492	0.33508	3.2725	0.1236	0.0096@ 0.0158	0.1401@ 0.0628	0.1371@ 0.1404	0.0705@ 0.2474

The phase influences the final pattern in the non-linear way. That means we cannot say anything regarding the final pattern taking into account only the evolution in the linear part when we are talking about the influence of the phase.
In order to prove the above-mentioned statements, we seed the network with the sum between mode 3 of amplitude 0.1 and phase pi/10 and mode 2 of amplitude 0.0905. The initial state, the evolution of the initial state to the final pattern and the final pattern are represented in the figure below.
From the table it can be easily seen that mode 2 has the biggest real part for the temporal eigenvalue. Despite this, mode 3 will “win” the competition in the non-linear part. This is because of the influence of the phase:

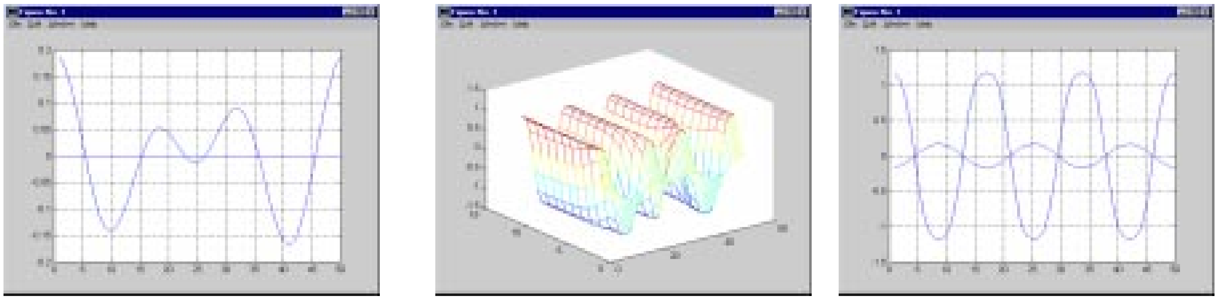

Fig. 3: Initial state, Evolution to the final pattern, final pattern

In the second experiment, we seed the network with mode 3 and 2 of the same amplitude. The phase of the mode 3 is now pi/54. The result is that mode 2 will win the competition for phase smaller or equal than this value:

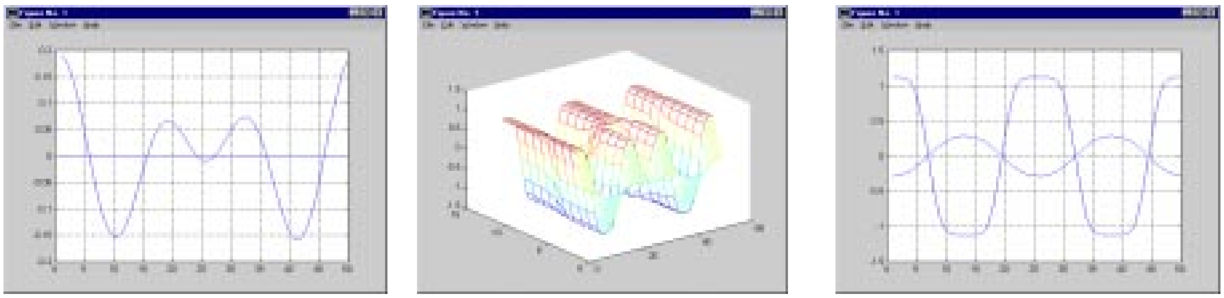

Fig. 4: Initial state, Evolution to the final pattern, final pattern

Moreover, the position of the “breaking points” in the cell’s characteristic does matter.

The left “breaking point” in the non-linear part of the cell’s characteristic is –1 and the right “breaking point” is changed from 1 to 10. The phase of the mode 3 is zero in the following experiment:

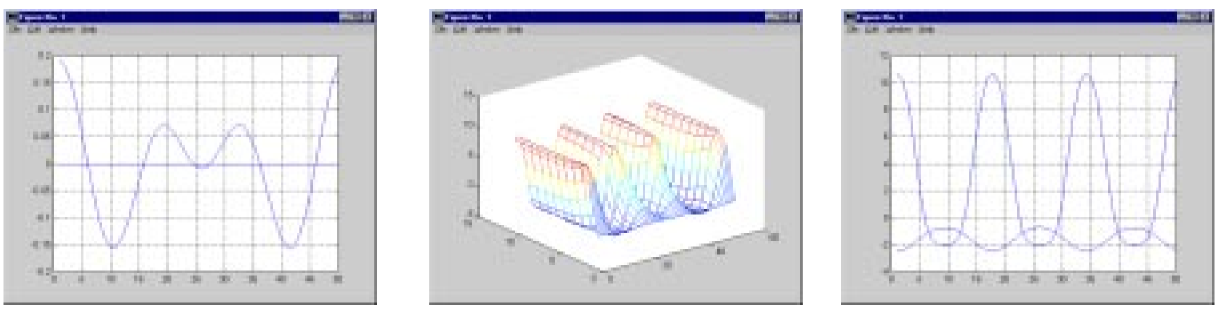

Fig. 5: Initial state, Evolution to the final pattern, final pattern

We change the phase to pi/5500. The result can be seen in the figure below:

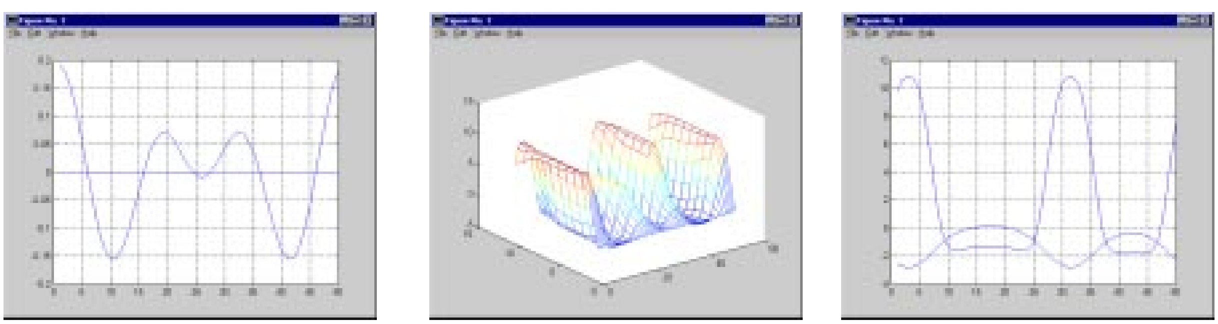

Fig. 6: Initial state, Evolution to the final pattern, final pattern

From the Fig. 6 (final pattern) it can be seen that mode 2 distorted “wins” the competition.

Then we eliminate the distortion of the winner (mode 2) by changing the phase of the mode 3 to pi/10:

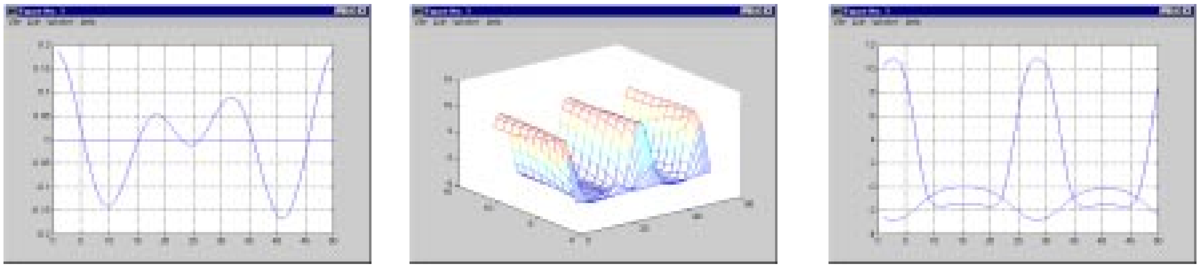

Fig. 7: Initial state, Evolution to the final pattern, final pattern

In the next four experiments, we will emphasize another important aspect: it is possible to obtain a new different pattern corresponding to a different mode from the mode(s) with which we seed the network. The importance of the phase will be stressed, too. We will seed the network with a sum between modes 1 and 4 with different amplitudes.

First, we use amplitude of 0.3 for mode 1 and 0.01 for mode 4. The phase is zero. Mode 2 will win the competition. Remark: in the initial spectrum composition: there wasn't mode 2 at all. This is up to the non-linearity.

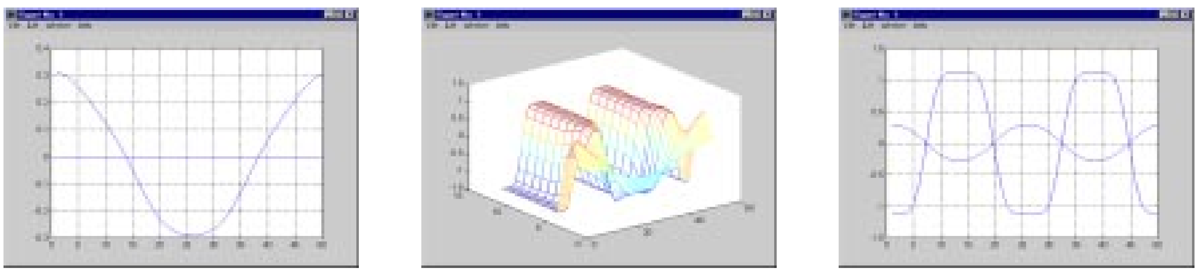

Fig. 8: Initial state, Evolution to the final pattern, final pattern

Then we change the amplitude of mode 1 to 0.2. The rest will be unchanged. Another mode that wasn't present in the initial spectrum wins the competition: mode 3:

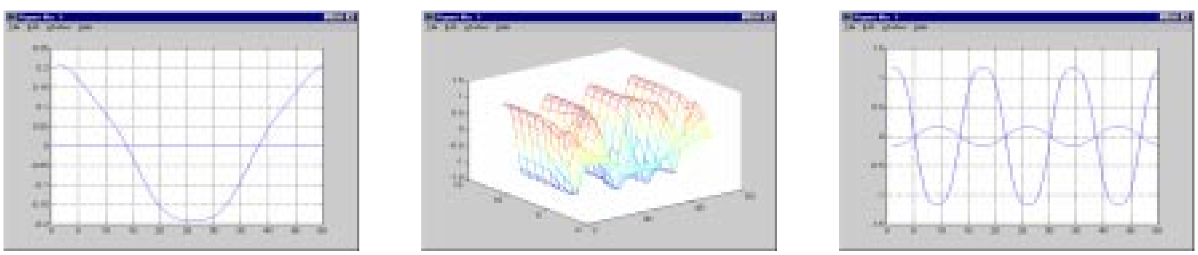

Fig. 9: Initial state, Evolution to the final pattern, final pattern

By slightly changing the amplitude of mode 1 to 0.1968, we obtain the pattern corresponding to the mode 4:

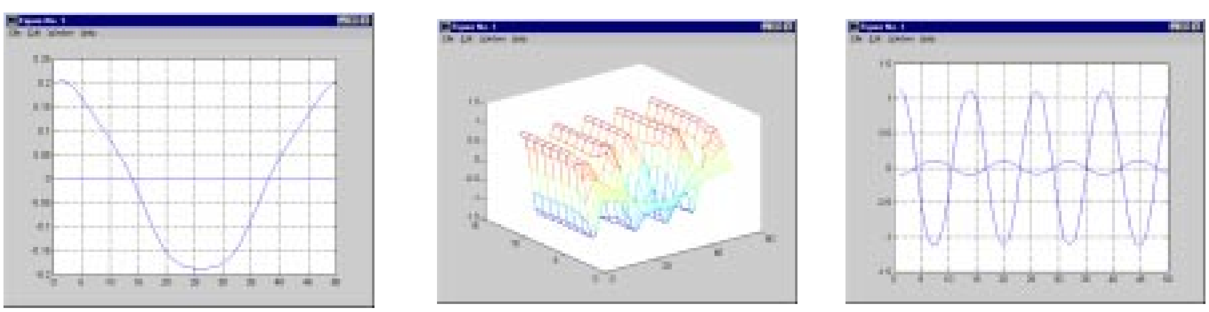

Fig. 10: Initial state, Evolution to the final pattern, final pattern

Mode 3 can be obtained by changing the phase of mode 4 from 0 to pi/10:

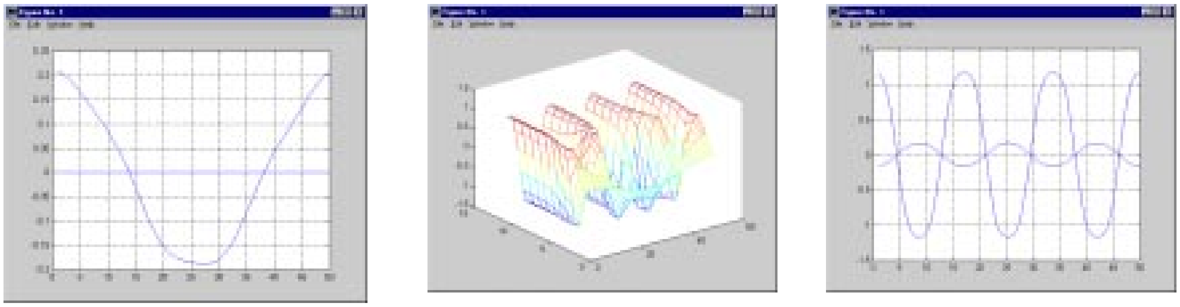

Fig. 11: Initial state, Evolution to the final pattern, final pattern

CONCLUDING REMARKS

In this work we have experimentally proved that the phase in a second order cell CNN can be crucial in the mode competition for Turing Pattern formation. For these situations the prediction of the final pattern according to the previous works based on mode decoupling techniques cannot be obtained. Moreover, new modes, other then the ones seeded into the network can appear as winners in the final pattern.

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