On the Oscillatory Behavior of Second Order Cell 1D CNN's

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Abstract **– Second order cell CNN's with first order neighborhoods have been thoroughly studied mainly for their capability to produce Turing patterns. In this communication it is shown that such systems may exhibit an oscillatory behavior as well. Using the decoupling technique valid for the central linear part of the cell characteristics, analytical conditions for the above oscillatory behavior have been derived.**

1 Introduction

Cellular Neural Networks made with second order cells were introduced [1-4] as special systems capable of producing Turing patterns. The main feature of such a behavior is that, in certain conditions, using stable identical cells, the network built through their identical connection could have the homogeneous equilibrium point unstable but exhibit nonhomogeneous stable equilibrium points (patterns).

Although the dynamics of CNN's is basically nonlinear, significant insight regarding their behavior has been obtained using a decoupling technique [ ] valid for the central linear part and related to the so-called *dispersion curve* representing the real part of the system's eigenvalues vs. spatial eigenvalues or vs. spatial modes. A necessary condition for pattern formation is that the dispersion curve has positive values for at least one spatial mode.

The cell parameters and the template ones characterize any CNN. Cell parameters will be used to shape the dispersion curve vs. spatial eigenvalues while the template parameters will select the position and width of a "window" under the dispersion. A typical dispersion curve vs. eigenvalues as well as vs. modes is represented in Fig.1. The behavior of the CNN depends on the part of the dispersion curve windowed according to the template coefficients.

An important aspect that has not been considered so far (in previous papers only the real part of the temporal eigenvalues was used) is the influence of the imaginary parts of the eigenvalues which characterize the oscillatory behavior of the spatial modes.

2 Theoretical background

In the following we consider two-port cells connected by means of two identical first order neighborhood templates. In the central linear part, the CNN is described by the following set of equations

$$\begin{cases} \dfrac{du_i(t)}{dt} = \gamma(f_u u_i + f_v v_i) + D_u O_{1D}(u_i) & i = 0..M-1 \\ \dfrac{dv_i(t)}{dt} = \gamma(g_u u_i + g_v v_i) + D_v O_{1D}(v_i) & \end{cases} \quad (1)$$

where f_u, f_v, g_u, g_v and γ characterize the cell resistive and capacitive parts D_u and D_v are scale factors and O_{1D} describes the template:

$$O_{1D}(u_i) = Cu_{i-1} + Bu_{i+1} + Au_i \quad (2)$$

which, in the following will be considered symmetric, i.e., B=C.

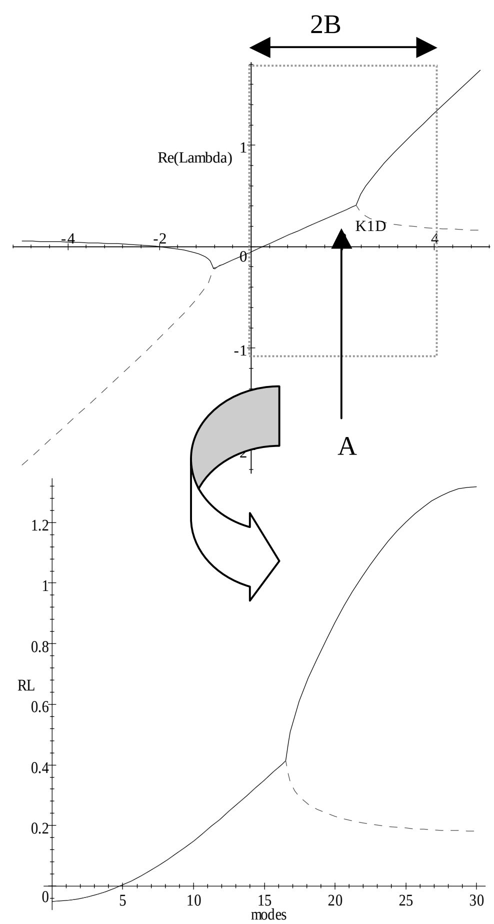

Figure 1 a) Dispersion curve vs. spatial eigenvalues b) Dispersion curve vs. modes obtained from a)

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Using the decoupling technique, based on the change of variable:

$$\begin{cases} u_i = \sum_{m=0}^{M-1} \Phi_M(m,i)\hat{u}_m & i = 0...M-1 \\ v_i = \sum_{m=0}^{M-1} \Phi_M(m,i)\hat{v}_m \end{cases} \quad (3)$$

and considering ring boundary conditions, the spatial eigenfunctions are of the form:

$$\Phi_M(\omega(m), \varphi(m), i) = e^{j(\omega(m)i+\varphi(m))} \quad (4)$$

where $\omega(m)=2\pi m/M$ and $\varphi(m)$ can have any value. It is easy to show that the corresponding spatial eigenvalues are:

$$K_{1D} = A + 2B\cos(\omega(m)) \quad (5)$$

and the dispersion curve for a CNN with different templates for the u and v layers is:

$$\mathrm{Re}\{\lambda_{1,2}(K_{1D},K'_{1D})\} = \mathrm{Re}\left[\gamma\frac{f_u+g_v}{2} + \frac{D_u K_{1D} + D_v K'_{1D}}{2} \pm \sqrt{\left[\gamma\frac{(g_v-f_u)}{2} - \frac{(D_u K_{1D} - D_v K'_{1D})}{2}\right]^2 + \gamma^2 f_v g_u}\right] \quad (6)$$

There are three zones in Fig. 1 corresponding to distinct real roots, and to complex conjugated roots (middle).

The above dispersion curve shows the possibility of various types of dynamics according to the shape of the curves and the position of the of the eigenvalues windows.

The relevant points of the dispersion curves for $K_{1D}=K_{1D}'$ are:

1. The extremities of the zone where the eigenvalues are complex conjugated:

$$K_{1D_{left}} = \frac{\gamma}{D_v - D_u}\left[(f_u - g_v) - 2\sqrt{-f_v g_u}\right]$$
$$K_{1D_{right}} = \frac{\gamma}{D_v - D_u}\left[(f_u - g_v) + 2\sqrt{-f_v g_u}\right] \quad (7)$$

2. The width of the middle zone:

$$\Delta K_{1D} = -\frac{4\gamma\sqrt{-f_v g_u}}{D_u - D_v} \quad (8)$$

3. The coordinates of the middle zone center:

$$K_{1Dmiddle} = -\frac{\gamma}{D_u - D_v}(f_u - g_v) \quad (9)$$

and

$$\mathrm{Re}\lambda(K_{1Dmiddle}) = \frac{\gamma}{2(D_u - D_v)}(g_v D_u - f_u D_v) \quad (10)$$

The necessary condition for the middle zone to exist is that the product $f_v g_u$ have negative sign so that K_{1Dleft} and $K_{1Dright}$ be real numbers. In the limit, when $f_v g_u=0$, there is only one value K_{1D} for which the temporal eigenvalues are complex conjugated.

4. When $D_u \neq -D_v$ there are two extreme points, a maximum and a minimum, each on the two branches of the dispersion curves.

$$K_{1D_1} = \frac{\gamma}{D_u - D_v}\left[g_v - f_u - (D_u + D_v)\sqrt{\frac{-f_v g_u}{D_u D_v}}\right] \quad (11)$$
$$K_{1D_2} = \frac{\gamma}{D_u - D_v}\left[g_v - f_u + (D_u + D_v)\sqrt{\frac{-f_v g_u}{D_u D_v}}\right]$$

The curve of the imaginary parts of the eigenvalues (when they exist) is:

$$\mathrm{Im}\{\lambda_{1,2}(K_{1D},K'_{1D})\} = \pm\sqrt{\left[\gamma\frac{(g_v-f_u)}{2} - \frac{(D_u K_{1D} - D_v K'_{1D})}{2}\right]^2 + \gamma^2 f_v g_u} \quad (12)$$

3. Oscillatory behavior

In the following we will consider a particular case of the dispersion curve characterized by a middle zone positioned on the horizontal axis as shown in Fig.2.

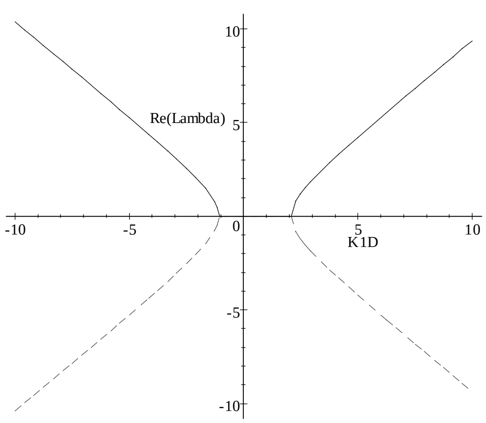

Figure 2: Dispersion curve for the particular case $\gamma=5$, $f_u=-0.1$, $f_v=-0.1$, $g_u=1$, $g_v=0.1$, $D_u=1$, $D_v=-1$, M=30

The imaginary part is represented in Figure 3.

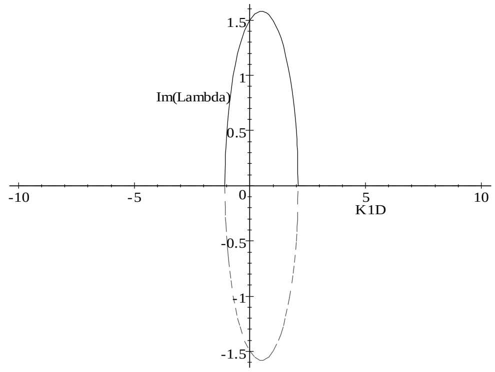

Figure 3: The imaginary parts of the temporal eigenvalues vs. spatial eigenvalues

Let us observe that, choosing part of the mode window in the horizontal part of the characteristic, these modes will be placed on the imaginary axis at values given by the curve in Fig. 3. As the center of the window is determined by A and its width by B, it is possible to chose the template so that all eigenvalus be on the imaginary axis. In this care, the spatial harmonics of the signal represented by the initial conditions will oscillate in time with frequencies determined by the imaginary part of the dispersion curve.

The imaginary part can be practically the same for all modes when the window width i.e., B is small. This corresponds to a practically horizontal part of the imaginary curve and, in the same time to weakly coupled cell. In particular, placing the center of the window at

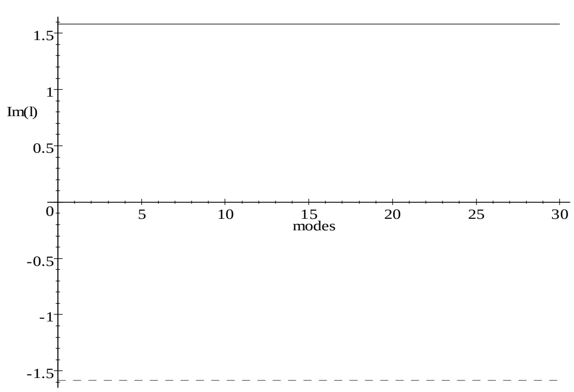

Figure 4: The imaginary part of the temporal eigenvalues for first situation (A=0.5, B=0.01)

$$K_{1Dmiddle} = -\frac{\gamma}{D_u - D_v}\left(f_u - g_v\right)$$ corresponding to the extreme values of the imaginary parts $\pm\gamma\sqrt{f_u g_v}$, the frequencies of oscillations can be almost identical.

Another case of interest is that when the eigenvalues have an important variation within the window. This situation can be obtained choosing the window towards the extremities of the middle part of the dispersion curve. The sign of B determines the order of the eigenvalues when passing from eigenvalues to modes. The representations of the imaginary parts of the temporal eigenvalues *vs. modes* for two cases is given in Figs. 5 and 6.

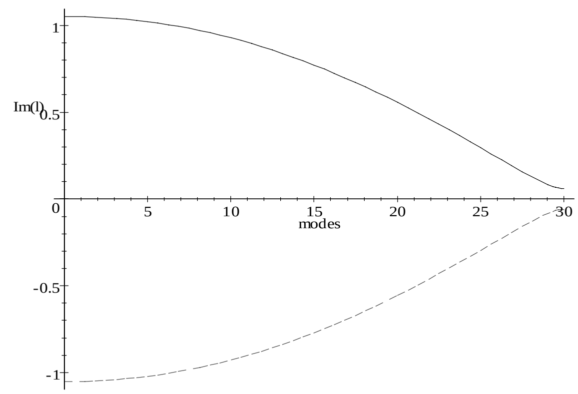

Figure 5: Imaginary parts of the temporal eigenvalues for A=-0.88 and B=0.1

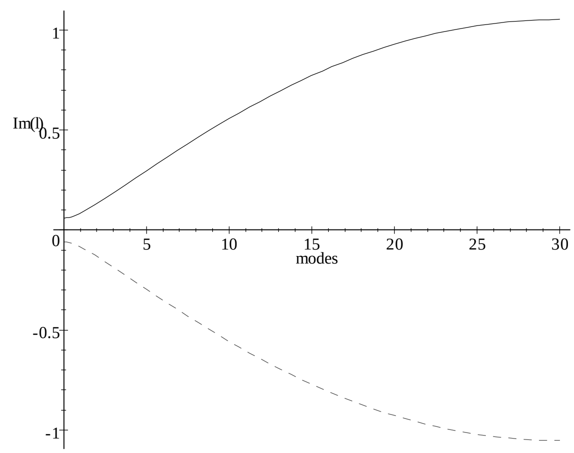

Figure 6: Imaginary parts of the temporal eigenvalues for A=-0.88 and B=-0.1

4. Simulation results

The simulation in Fig. 7 corresponds to temporal eigenvalues almost identical, as in Fig. 4. The initial conditions were the sum of modes 3 and 13, both with amplitude 0.1.

The simulation in Fig. 8 corresponds to distinct temporal eigenvalues (see Fig. 5). The initial conditions were the sum of the spatial modes 5 and 13 with amplitudes equal to 0.1. It is apparent that, in this case, there is no synchronization between the two modes.

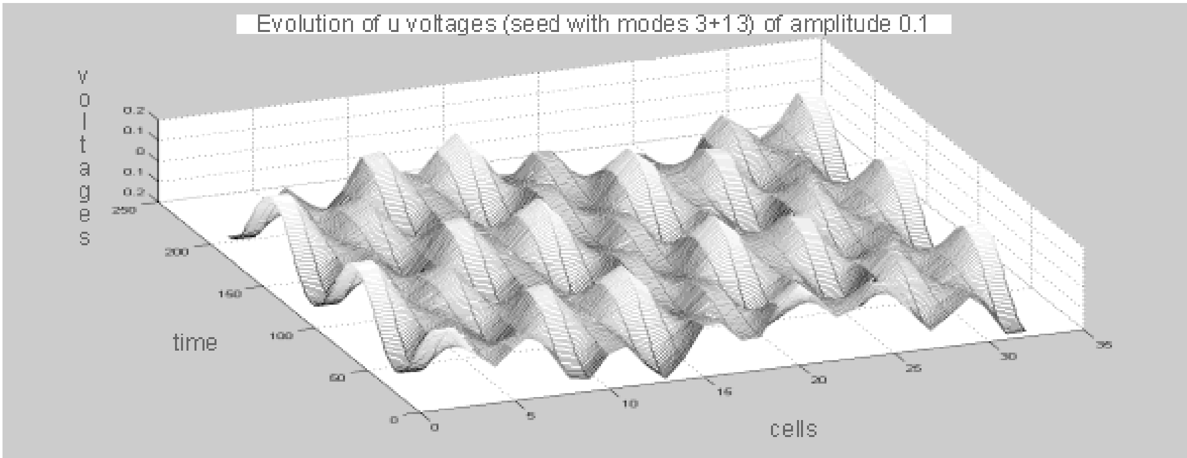

Figure 7: Time evolution of the pattern for system corresponding to the dispersion curve represented in Fig. 4

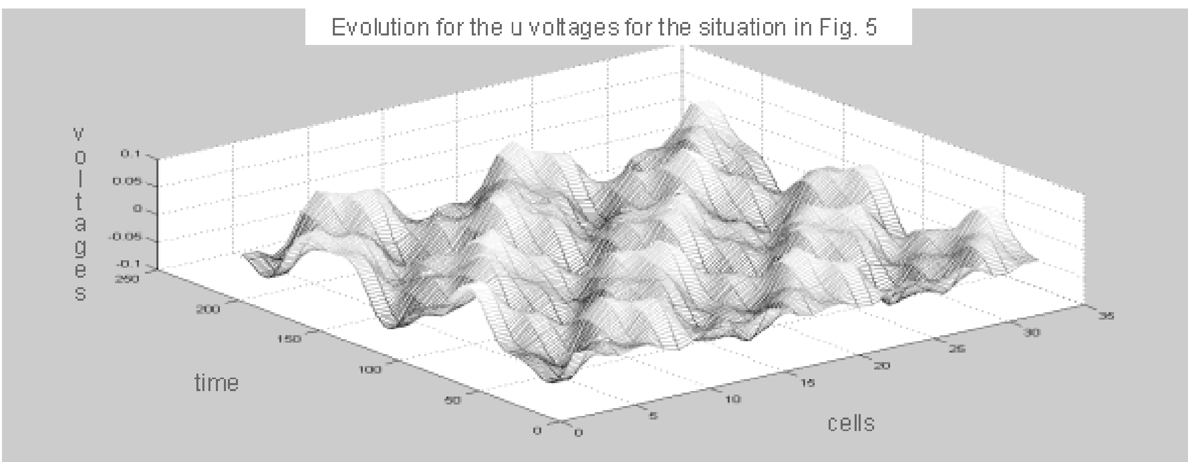

Figure 8: Time evolution of the pattern for system corresponding to the dispersion curve represented in Fig. 5

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