Low Passband Sensitivity FIR Digital Filter

- We consider here the Type 1 filter as it is the most general linear-phase filter and can realize any type of frequency response
- The frequency response of a Type 1 FIR transfer function $H(z)$ of order N can be expressed as

$$H(e^{j\omega}) = e^{-j\omega N/2}\breve{H}(\omega)$$

where $\breve{H}(\omega)$, a real function of ω, is its amplitude response

1

Low Passband Sensitivity FIR Digital Filter

- If $H(z)$ is a BR function, then $\breve{H}(\omega) \le 1$
- Its delay-complementary transfer function $G(z)$ defined by

$$G(z) = z^{-N/2} - H(z)$$

has a frequency response given by

$$G(e^{j\omega}) = e^{-j\omega N/2}[1 - \breve{H}(\omega)] = e^{-j\omega N/2}\breve{G}(\omega)$$

where $\breve{G}(\omega) = 1 - \breve{H}(\omega)$ is its amplitude response

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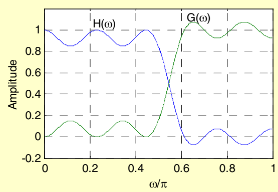
Low Passband Sensitivity FIR Digital Filter

- Amplitude responses of a typical delay-complementary FIR filter pair are shown below



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Low Passband Sensitivity FIR Digital Filter

- It follows from the plots of the amplitude responses that at $\omega = \omega_k$, where $|H(e^{j\omega_k})| = 1$, $\breve{G}(\omega)$ has double zeros
- Thus, $G(z)$ can be expressed as

$$G(z) = G_a(z)\prod_{k=1}^{L}(1 - 2\cos\omega_k z^{-1} + z^{-2})^2$$
$$= G_a(z)G_b(z)$$

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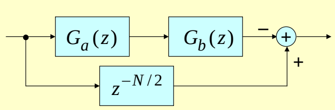
Low Passband Sensitivity FIR Digital Filter

- A delay-complementary realization of $H(z)$ based on $H(z) = z^{-N/2} - G(z)$ is shown below



- $G_b(z)$ consists of L 4-th order FIR sections with the k-th section having a transfer function $(1 - 2\cos\omega_k z^{-1} + z^{-2})^2$

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Low Passband Sensitivity FIR Digital Filter

- If the multiplier coefficient $2\cos\omega_k$ of the k-th section is quantized, its zeros are still double and remain on the unit circle
- Thus, quantization of the coefficients of $G_b(z)$ does not change the sign of the amplitude response $\breve{G}(\omega)$, and in the passband of $H(z)$, $\breve{G}(\omega) \ge 0$

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Low Passband Sensitivity FIR Digital Filter

- In addition, $G_a(z)$ has no zeros on the unit circle, and quantization of its coefficients also does not affect the sign of $\breve{G}(\omega)$
- Hence, $\breve{H}(\omega)$ continues to remain bounded above by unity
- ➡ The realization of $H(z)$ as indicated remains structurally BR or structurally passive with regard to all coefficients, resulting in a low passband sensitivity realization

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Low Passband Sensitivity FIR Digital Filter

- Example - The filter specifications are length 13 with a normalized passband edge at 0.5 and a normalized stopband edge at 0.6 with equal weights to passband and stopband ripples
- Using the M-file `remez` we determine the transfer function of the lowpass filter $H(z)$ and form its delay-complementary filter

$$G(z) = z^{-6} - H(z)$$

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Low Passband Sensitivity FIR Digital Filter

- $G(z)$ has 6 zeros on the unit circle: 2 zeros at $z = 1$, a pair of complex conjugate zeros at $z = -0.26463064626566 \pm j0.9643498437$ and a pair of complex conjugate zeros at $z = -0.27683551142484 \pm j0.96091732945$
- These unit circle zeros constitute

$$G_b(z) = (1 - z^{-1})^2 (1 - 0.52926129z^{-1} + z^{-2}) \times (1 - 0.5536710228497z^{-1} + z^{-2})$$

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Low Passband Sensitivity FIR Digital Filter

- By factoring out $G_b(z)$ from $G(z)$ we get

$$G_a(z) = 0.04107997 + 0.051971544z^{-1} - 0.12094731168z^{-2} - 0.30704562224z^{-3} + 0.120947311687z^{-4} - 0.0.051971544z^{-5} + 0.04107997195619z^{-6}$$

- Next we quantize the coefficients of $G_a(z)$ and $G_b(z)$ by rounding the fractional part to 2 decimal digits

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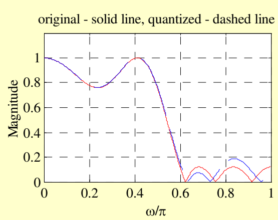
Low Passband Sensitivity FIR Digital Filter

- Finally, from $G(z)$ with quantized coefficients, the delay-complementary transfer function $H(z)$ is determined



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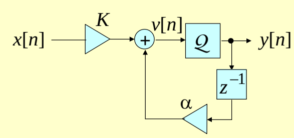
First-Order Error-Feedback Structure

- Consider the scaled first-order section



- We assume that all multiplier coefficients are signed $(b + 1)$-bit fractions
- The quantization error signal is given by

$$e[n] = y[n] - v[n]$$

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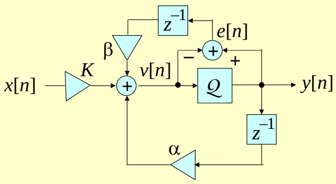
First-Order Error-Feedback Structure

- The first-order section is modified by feeding back the error signal $e[n]$ to the system through a delay and a multiplier β as shown below



First-Order Error-Feedback Structure

- In practice, β is chosen to be a simple integer or a power-of-2 fraction, such as ± 1, ± 2, or ± 0.5 so that the multiplication can be performed using a shift operation and will not introduce an additional quantization error

First-Order Error-Feedback Structure

- Analyzing the error-feedback structure we arrive at its transfer function

$$H(z) = \left.\frac{Y(z)}{X(z)}\right|_{E(z)=0} = \frac{K}{1-\alpha z^{-1}}$$

- The noise transfer function $G(z)$ with the error feedback, with $y[n]$ as the output is given by

$$G(z) = \left.\frac{Y(z)}{E(z)}\right|_{X(z)=0} = \frac{1+\beta z^{-1}}{1-\alpha z^{-1}}$$

First-Order Error-Feedback Structure

- The noise transfer function without the error feedback (β = 0) is given by

$$G_0(z) = \frac{1}{1-\alpha z^{-1}}$$

- The output noise variance of the error-feedback structure is given by

$$\sigma_\gamma^2 = \left(\frac{1+2\alpha\beta+\beta^2}{1-\alpha^2}\right)\sigma_o^2$$

where σ_o^2 is the variance of $e[n]$

First-Order Error-Feedback Structure

- σ_γ^2 is a minimum when $\beta = -\alpha$
- However, in practice $|\alpha| < 1$
- Hence $\beta = -\alpha$ will introduce an additional quantization noise source, making the analysis resulting in the expression for σ_γ^2 invalid
- Thus, β should be chosen as an integer with a value close to that of $-\alpha$

First-Order Error-Feedback Structure

- For $|\alpha| < 0.5$, β = 0, implying no error feedback
- However, in this case, the pole of $H(z)$ is far from the unit circle, and as a result, the output noise variance σ_γ^2 is not that high
- For $|\alpha| \geq 0.5$, choose $\beta = (-1)\,\mathrm{sgn}(\alpha)$
- Using this value of β we get

$$\sigma_\gamma^2 = \frac{2}{1+|\alpha|}\sigma_o^2$$

First-Order Error-Feedback Structure

- The output noise variance with $\beta = 0$ is

$$\sigma_\gamma^2 = \frac{1}{1-\alpha^2}\sigma_o^2$$

- Thus, error feedback has increased the SNR by a factor of

$$-10\log_{10}[2(1-|\alpha|)]\,\text{dB}$$

- This increase in SNR is quite significant if the pole is closer to the unit circle

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First-Order Error-Feedback Structure

- For example if $|\alpha| = 0.99$, the improvement is about 17 dB, which is equivalent to about 3 bits of increased accuracy compared to the case without error feedback
- Additional hardware requirements for the error-feedback structure are two new adders and an additional storage register

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First-Order Error-Feedback Structure

- The noise transfer function for the error-feedback structure can be expressed as

$$G(z) = (1+\beta z^{-1})G_0(z)$$

where $G_0(z)$ is the noise transfer function without error feedback

- The error-feedback circuit is **shaping** the error spectrum by modifying the input quantization noise $E(z)$ to

$$E_s(z) = (1+\beta z^{-1})E(z)$$

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First-Order Error-Feedback Structure

- The output noise is generated by passing $E_s(z)$ through the usual noise transfer function $G_0(z)$
- To illustrate the effect of noise spectrum shaping, consider the case of a narrow-band lowpass first-order filter with $\alpha \to 1$
- We choose $\beta = -1$ and as a result $E_s(z)$ has a zero at $z = 1$ ($\omega = 0$)

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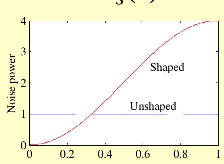
First-Order Error-Feedback Structure

- The power spectral density of the unshaped quantization noise $E(z)$ is σ_o^2, a constant
- The power spectral density of the shaped quantization noise $E_s(z)$ is $4\sin^2(\omega/2)\sigma_o^2$



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First-Order Error-Feedback Structure

- The noise shaping redistributes the noise so as to move it mostly into the stopband of the lowpass filter, thus reducing the noise variance
- Because of the noise redistribution caused by the error-feedback, this approach has also been called the **error spectrum shaping method**

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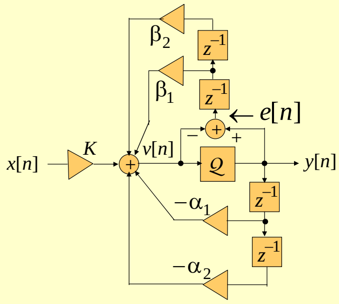
Second-Order Error-Feedback Structure



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Second-Order Error-Feedback Structure

- The noise transfer function is given by

$$G(z) = \frac{1+\beta_1 z^{-1} + \beta_2 z^{-2}}{1+\alpha_1 z^{-1} + \alpha_2 z^{-2}}$$

- The output round-off noise variance for $\mathcal{L}_2$-scaling is given by

$$\sigma_\gamma^2 = (\|G\|_2)^2 \sigma_o^2$$

- A choice of $\beta_1 = \alpha_1$ and $\beta_2 = \alpha_2$ makes $\|G\|_2 = 1$, yielding $\sigma_\gamma^2 = \sigma_o^2$, an apparent optimal solution

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Second-Order Error-Feedback Structure

- However, this choice for the multiplier coefficients in the error-feedback path introduces additional quantization noise sources that invalidates the expression for σ_γ^2
- A more attractive solution is to make β_1 and β_2 integers with values close to α_1 and α_2, respectively

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Second-Order Error-Feedback Structure

- For example, for a narrow-band lowpass transfer function, the poles are close to the unit circle and to the real axis, i.e., $r \approx 1$ and $\theta \approx 0$
- Then, α_1 is close to -2 and α_2 is close to 1
- In this case, choose $\beta_1 = -2$ and $\beta_2 = 1$
- Then

$$G(z) = \frac{1-2z^{-1} + z^{-2}}{1+\alpha_1 z^{-1} + \alpha_2 z^{-2}}$$

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Second-Order Error-Feedback Structure

- For a very narrowband lowpass filter with $r = 0.995$, $\theta = 0.07\pi$, and $b = 16$, the second-order error-feedback structure has an SNR that is approximately 25 dB higher than that without the error feedback
- The second-order error-feedback structure also provides a noise shaping

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Second-Order Error-Feedback Structure

- The error-feedback circuit shapes the error spectrum by modifying the input quantization noise $E(z)$ to

$$E_s(z) = (1-z^{-1})^2 E(z)$$

- The output noise is generated by passing $E_s(z)$ through the usual noise transfer function

$$G_0(z) = \frac{1}{1+\alpha_1 z^{-1} + \alpha_2 z^{-2}}$$

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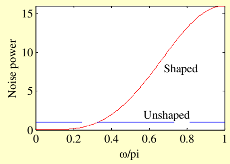
Second-Order Error-Feedback Structure

- The power spectral density of the shaped noise source $E_s(z)$ is $16\sin^4(\omega/2)\sigma_o^2$
- The power spectral density of the unshaped noise source is σ_o^2



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Limit Cycles in IIR Digital Filters

- So far we have treated the analysis of finite wordlength effects using a linear model of the system
- A practical digital filter is a nonlinear system caused by the quantization of the arithmetic operations
- Such nonlinearities may cause an IIR filter, which is stable under infinite precision, to exhibit an unstable behavior under finite precision arithmetic for specific input signals

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Limit Cycles in IIR Digital Filters

- This type of instability usually results in an oscillatory periodic output called a **limit cycle**
- The system remains in this condition until an input of sufficiently large amplitude is applied to move the system into a more conventional operation

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Limit Cycles in IIR Digital Filters

- Limit cycles occur in IIR filters due to the presence of feedback
- Such oscillations are absent in FIR filters as they do not have any feedback path
- There are two types of limit cycles

  (1) **Granular limit cycle** is usually of low amplitude

  (2) **Overflow limit cycle** has large amplitudes

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Limit Cycles in IIR Digital Filters

- Two types of granular limit cycles have been observed in IIR digital filters:

  (1) **Inaccessible limit cycle** - can appear only if the initial conditions of the digital filter at the time of starting pertain to that limit cycle

  (2) **Accessible limit cycle** - can appear by starting the digital filter with initial conditions not pertaining to the limit cycle

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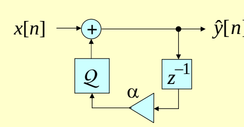
Granular Limit Cycles

- Consider the first-order IIR filter as shown below



- Assume the quantization operation to be rounding and the filter to be implemented with a signed 6-bit fractional arithmetic
- The nonlinear difference equation characterizing the filter is given by

$$\hat{y}[n] = Q(\alpha \cdot \hat{y}[n-1]) + x[n]$$

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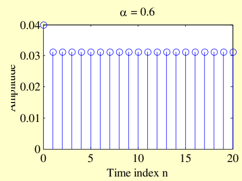
Granular Limit Cycles

- For $x[n] = 0.04\delta[n]$, $\hat{y}[-1] = 0$, and $\alpha = 0.6$, the output of the filter is as shown below



- The limit cycle generated has a period of 1

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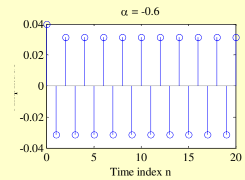
Granular Limit Cycles

- For $x[n] = 0.04\delta[n]$, $\hat{y}[-1] = 0$, and $\alpha = -0.6$ the output of the filter is as shown below



- The limit cycle generated has a period of 2

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Overflow Limit Cycles

- Limit-cycle-like oscillations can also result from overflow in digital filters implemented with finite precision arithmetic
- The amplitude of the overflow oscillations can cover the whole dynamic range of the register experiencing the overflow
- Overflow limit cycles are thus much more serious in nature than the granular limit cycles

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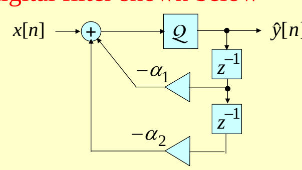
Overflow Limit Cycles

- Consider the causal all-pole second-order IIR digital filter shown below



- Assume implementation using sign-magnitude 4-bit arithmetic with a rounding of the sum of products by a single quantizer

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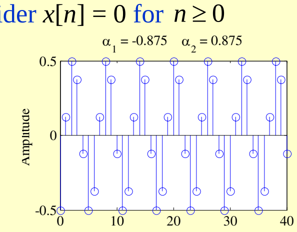
Overflow Limit Cycles

- Let $\alpha_1 = -0.875$, $\alpha_2 = 0.875$, $\hat{y}[-1] = -0.625$ and $\hat{y}[-2] = -0.125$
- Consider $x[n] = 0$ for $n \geq 0$



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Overflow Limit Cycles

- The second-order direct form IIR structure with multiplier coefficients α_1 and α_2 remains stable if $|\alpha_2| < 1$ and $|\alpha_1| < 1 + \alpha_2$
- However, the structure can still get into a zero-input overflow oscillation mode for a large range of values of the filter constants satisfying the stability constraint when implemented using two's-complement arithmetic with rounding

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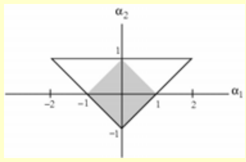
Overflow Limit Cycles

- It has been shown that overflow limit cycles under zero-input cannot occur if the filter coefficients lie in the shaded region inside the stability triangle shown below



Limit Cycle Free Structures

- Conditions for a digital filter structure to not support limit cycles have been derived in terms of its state transition matrix
- For a second-order causal LTI digital filter, the state-space representation relating the output $y[n]$ to the input $x[n]$ is given by

$$\begin{bmatrix} s_1[n+1] \\ s_2[n+1] \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} s_1[n] \\ s_2[n] \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} x[n]$$

$$y[n] = [c_1 \quad c_2] \begin{bmatrix} s_1[n] \\ s_2[n] \end{bmatrix} + d\, x[n]$$

Limit Cycle Free Structures

- Let $\quad \mathbf{s}[n] = [s_1[n] \quad s_2[n]]^T$

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \mathbf{B} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}, \mathbf{C} = [c_1 \quad c_2]$$

- The state-space description is then compactly written as

$$\mathbf{s}[n+1] = \mathbf{A}\,\mathbf{s}[n] + \mathbf{B}\,x[n]$$
$$y[n] = \mathbf{C}\,\mathbf{s}[n] + d\,x[n]$$

- **A** is called the **state-transition matrix**
- **s**$[n]$ is called the **state-vector**

Limit Cycle Free Structures

- The quantization errors caused by the quantization of the state-transition equation

$$\mathbf{s}[n+1] = \mathbf{A}\,\mathbf{s}[n] + \mathbf{B}\,x[n]$$

  go through the feedback loop and are responsible for the generation of limit cycles
- Assume $s_1[n+1]$ and $s_2[n+1]$ are quantized
- Delayed versions of these quantized signals are $s_1[n]$ and $s_2[n]$

Limit Cycle Free Structures

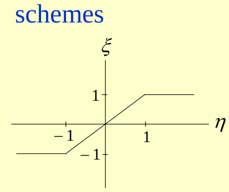
- A quantizer is defined to be **passive** if

$$|\mathcal{Q}(x)| \le |x|, \quad \text{for all } x$$

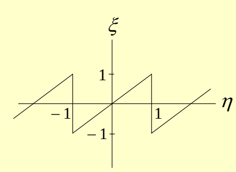
- If x is inside the dynamic range of the system, then for magnitude truncation above inequality holds
- If x is outside the dynamic range, for example by overflow, it must be brought back to the range by following the schemes discussed next

Handling Overflow

- If η, the sum of two fixed-point fractions, exceeds the dynamic range $[-1, 1)$, it is substituted with a number ξ which is within the range using one of the two following schemes



Saturation overflow



Two's-complement overflow

Limit Cycle Free Structures

- Thus, magnitude truncation followed by one of the two overflow handling schemes is again a passive quantizer
- A digital filter structure with a state transition matrix satisfying

$$\mathbf{A}^T\mathbf{A} = \mathbf{A}\mathbf{A}^T$$

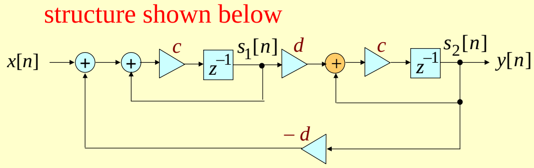
- has been called a **normal form structure**

Limit Cycle Free Structures

- A normal form structure with passive quantizers does not support zero-input limit cycles of either type
- The state transition matrix **A** satisfying the condition $\mathbf{A}^T\mathbf{A} = \mathbf{A}\mathbf{A}^T$ and $\|\mathbf{A}\|_2 < 1$ is called a **normal matrix**

Limit Cycle Free Structures

- Example - Consider the digital filter structure shown below



- Analysis yields

$$s_1[n+1] = c\,s_1[n] - cd\,s_2[n] + cx[n]$$
$$s_2[n+1] = cd\,s_1[n] + c\,s_2[n]$$

Limit Cycle Free Structures

- The state transition matrix is given by

$$\mathbf{A} = \begin{bmatrix} c & -cd \\ cd & c \end{bmatrix}$$

- The transfer function of the structure is

$$H(z) = \frac{c^2 d z^{-2}}{1 - 2cz^{-1} + c^2(1+d^2)z^{-2}}$$

Limit Cycle Free Structures

- Comparing the denominator of $H(z)$ with that of a second-order IIR transfer function with poles at $z = re^{\pm j\theta}$ (with $r < 1$ for stability) we obtain $c = r\cos\theta$ and $d = \tan\theta$
- Thus

$$A = \begin{bmatrix} r\cos\theta & -r\sin\theta \\ r\sin\theta & r\cos\theta \end{bmatrix}$$

- Note: $\mathbf{A}^T\mathbf{A} = \mathbf{A}\mathbf{A}^T = r^2\mathbf{I}$ and $\|\mathbf{A}\|_2 = r < 1$
- ⟹ The filter is a **normal form** structure