IIR Filter structures

ELG6163

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Objective

- Stuctures
  - Direct form
  - Transposed direct form
  - Lattice-ladder form
  - Parallel realization
  - Cascade realization
  - Bi-quad coupled realization
  - State space realization
- Implementation

Basic IIR Digital Filter Structures

- The causal IIR digital filters we are concerned with in this course are characterized by a real rational transfer function of z^{-1} or, equivalently by a constant coefficient difference equation
- From the difference equation representation, it can be seen that the realization of the causal IIR digital filters requires some form of feedback

Basic IIR Digital Filter Structures

- An N-th order IIR digital transfer function is characterized by $2N+1$ unique coefficients, and in general, requires $2N+1$ multipliers and $2N$ two-input adders for implementation
- **Direct form IIR filters**: Filter structures in which the multiplier coefficients are precisely the coefficients of the transfer function

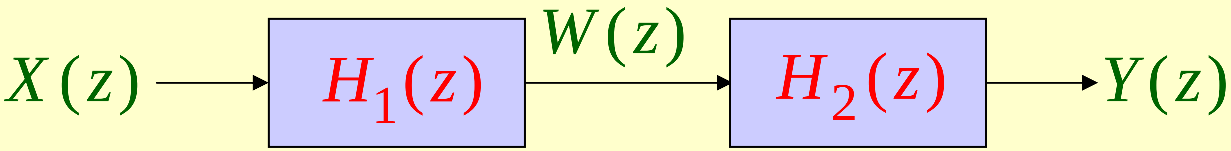
Direct Form IIR Digital Filter Structures

- Consider for simplicity a 3rd-order IIR filter with a transfer function

$$H(z) = \frac{P(z)}{D(z)} = \frac{p_0 + p_1 z^{-1} + p_2 z^{-2} + p_3 z^{-3}}{1 + d_1 z^{-1} + d_2 z^{-2} + d_3 z^{-3}}$$

- We can implement $H(z)$ as a cascade of two filter sections as shown on the next slide

Direct Form IIR Digital Filter Structures



where

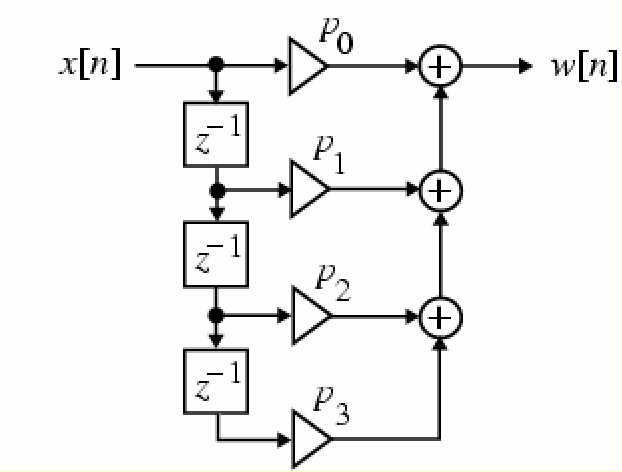
$$H_1(z) = \frac{W(z)}{X(z)} = P(z) = p_0 + p_1 z^{-1} + p_2 z^{-2} + p_3 z^{-3}$$

$$H_2(z) = \frac{Y(z)}{W(z)} = \frac{1}{D(z)} = \frac{1}{1 + d_1 z^{-1} + d_2 z^{-2} + d_3 z^{-3}}$$

Direct Form IIR Digital Filter Structures

- The filter section $H_1(z)$ can be seen to be an FIR filter and can be realized as shown below

$$w[n] = p_0 x[n] + p_1 x[n-1] + p_2 x[n-2] + p_3 x[n-3]$$

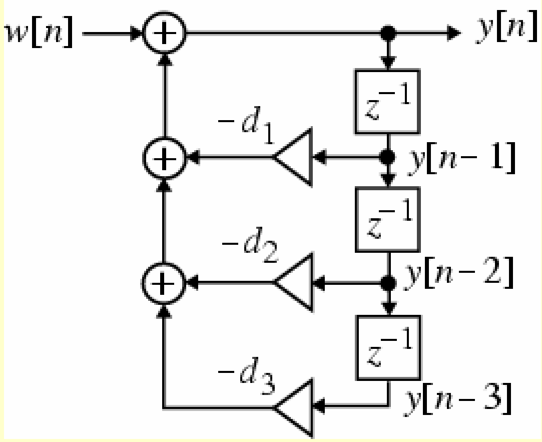


Direct Form IIR Digital Filter Structures

- The time-domain representation of $H_2(z)$ is given by

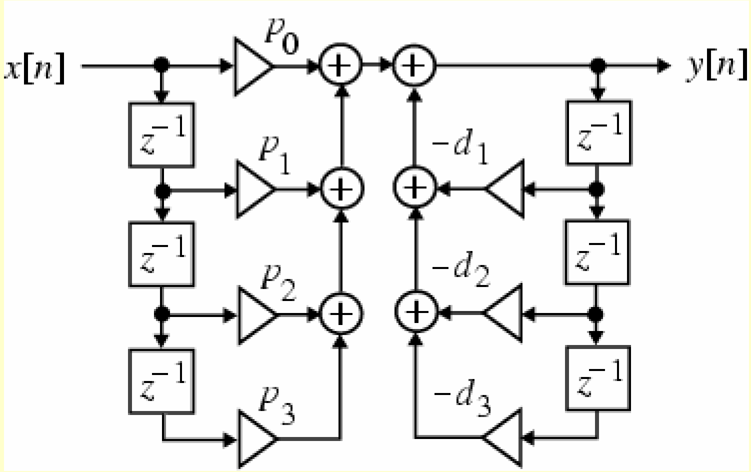
$$y[n] = w[n] - d_1 y[n-1] - d_2 y[n-2] - d_3 y[n-3]$$

Realization of $H_2(z)$ follows from the above equation and is shown on the right



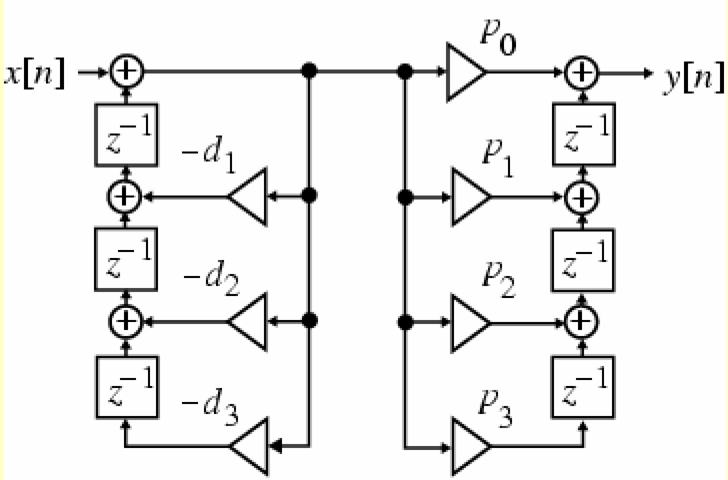
Direct Form IIR Digital Filter Structures

- A cascade of the two structures realizing $H_1(z)$ and $H_2(z)$ leads to the realization of $H(z)$ shown below and is known as the **direct form I** structure



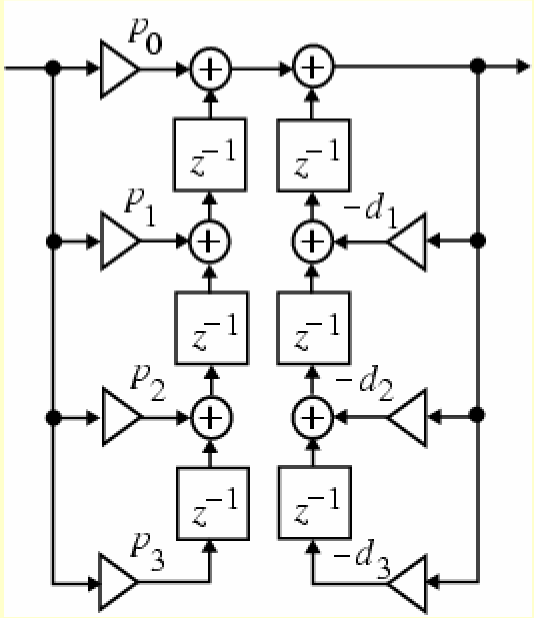
Direct Form IIR Digital Filter Structures

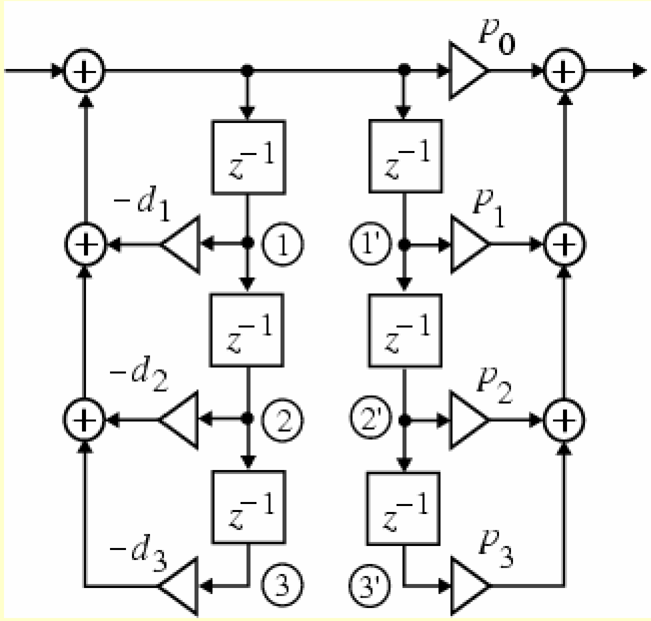
- Note: The direct form I structure is noncanonic as it employs 6 delays to realize a 3rd-order transfer function
- A transpose of the direct form I structure is shown on the right and is called the **direct form I$_t$ structure**



Direct Form IIR Digital Filter Structures

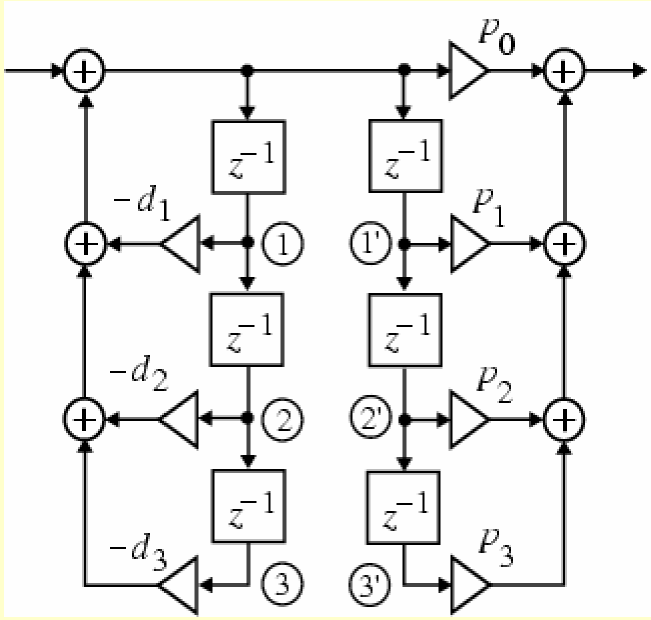
- Various other noncanonic direct form structures can be derived by simple block diagram manipulations as shown below





Direct Form IIR Digital Filter Structures

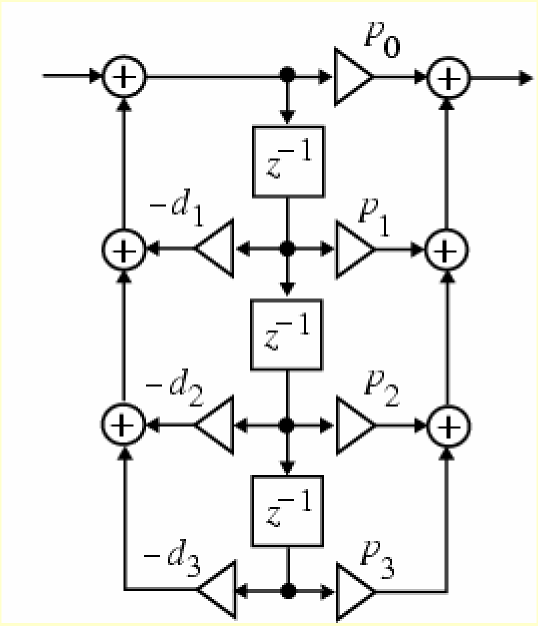
- Observe in the direct form structure shown below, the signal variable at nodes 1 and 1' are the same, and hence the two top delays can be shared



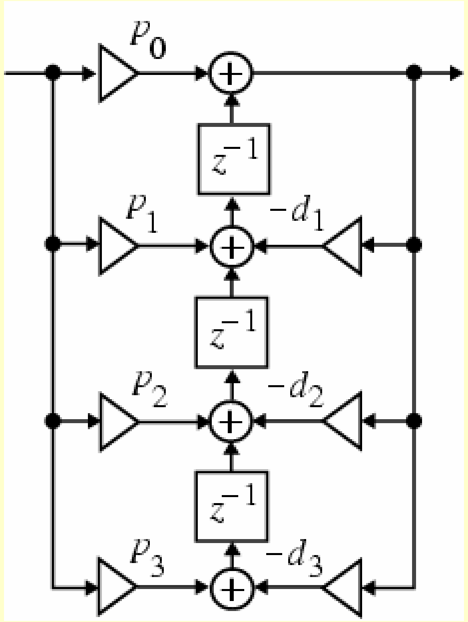
Direct Form IIR Digital Filter Structures

- Likewise, the signal variables at nodes 2 and 2' are the same, permitting the sharing of the middle two delays
- Following the same argument, the bottom two delays can be shared
- Sharing of all delays reduces the total number of delays to 3 resulting in a canonic realization shown on the next slide along with its transpose structure

Direct Form IIR Digital Filter Structures



Direct Form II



Direct Form II$_t$

- Direct form realizations of an *N*-th order IIR transfer function should be evident

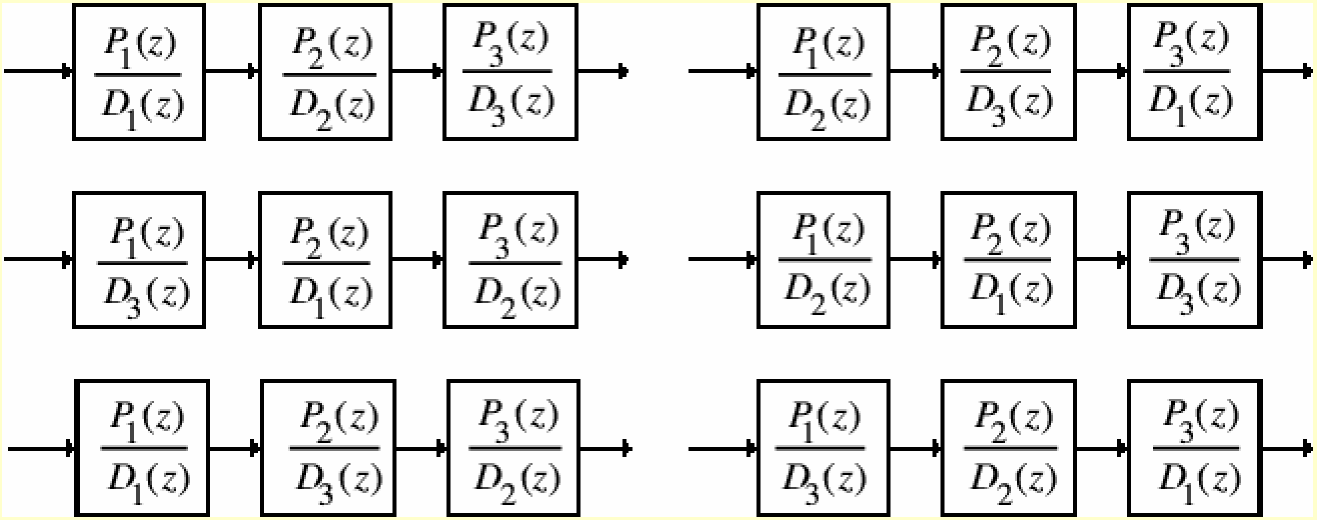
Cascade Form IIR Digital Filter Structures

- By expressing the numerator and the denominator polynomials of the transfer function as a product of polynomials of lower degree, a digital filter can be realized as a cascade of low-order filter sections
- Consider, for example, $H(z) = P(z)/D(z)$ expressed as

$$H(z) = \frac{P(z)}{D(z)} = \frac{P_1(z)P_2(z)P_3(z)}{D_1(z)D_2(z)D_3(z)}$$

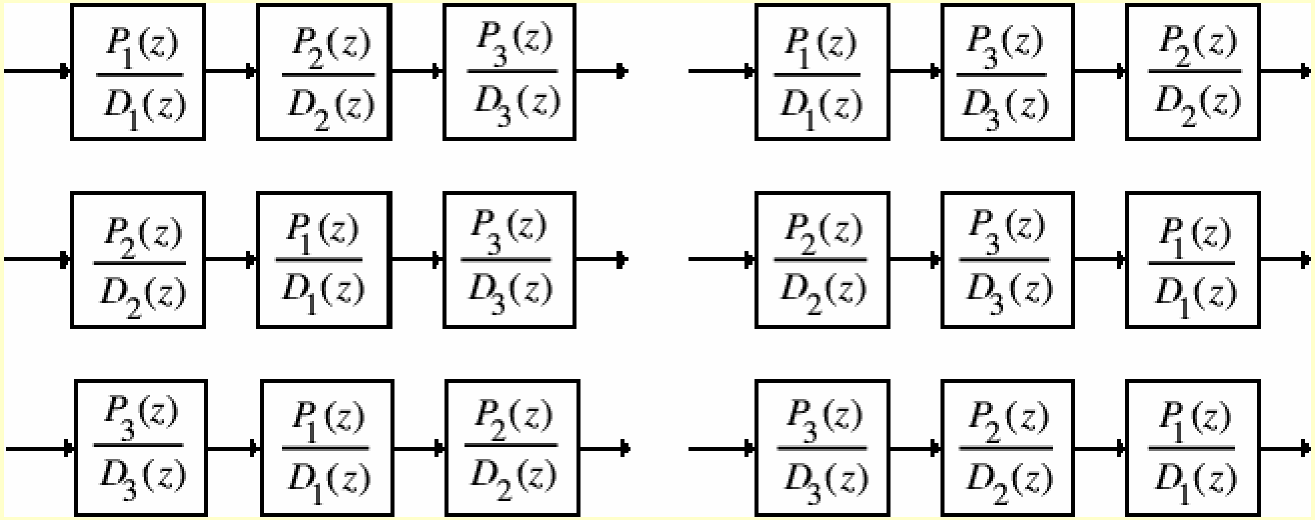
Cascade Form IIR Digital Filter Structures

- Examples of cascade realizations obtained by different pole-zero pairings are shown below



Cascade Form IIR Digital Filter Structures

- Examples of cascade realizations obtained by different ordering of sections are shown below



Cascade Form IIR Digital Filter Structures

- There are altogether a total of 36 different cascade realizations of
$$H(z) = \frac{P_1(z)P_2(z)P_2(z)}{D_1(z)D_2(z)D_3(z)}$$
based on pole-zero-pairings and ordering
- Due to finite wordlength effects, each such cascade realization behaves differently from others

Cascade Form IIR Digital Filter Structures

- Usually, the polynomials are factored into a product of 1st-order and 2nd-order polynomials:

$$H(z) = p_0 \prod_k \left(\frac{1+\beta_{1k} z^{-1} + \beta_{2k} z^{-2}}{1+\alpha_{1k} z^{-1} + \alpha_{2k} z^{-2}} \right)$$

- In the above, for a first-order factor

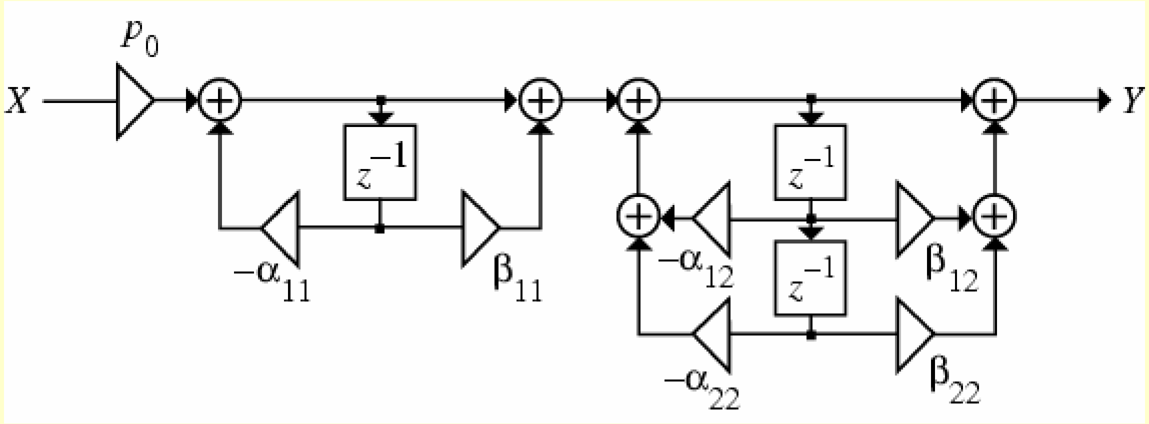
$$\alpha_{2k} = \beta_{2k} = 0$$

Cascade Form IIR Digital Filter Structures

- Consider the 3rd-order transfer function

$$H(z) = p_0 \left(\frac{1+\beta_{11} z^{-1}}{1+\alpha_{11} z^{-1}} \right) \left(\frac{1+\beta_{12} z^{-1} + \beta_{22} z^{-2}}{1+\alpha_{12} z^{-1} + \alpha_{22} z^{-2}} \right)$$

- One possible realization is shown below



Cascade Form IIR Digital Filter Structures

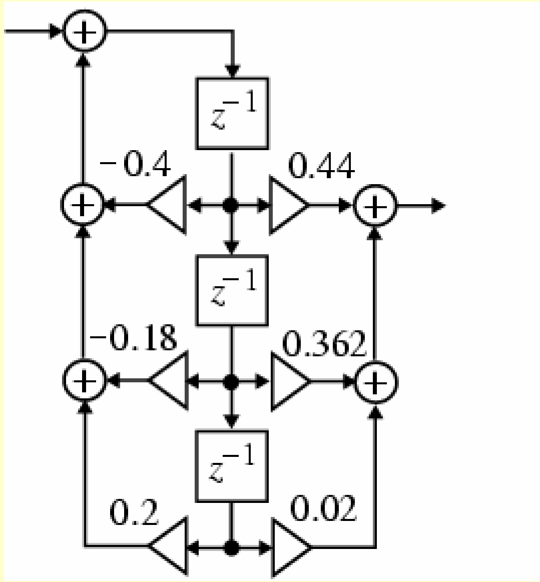
- Example - Direct form II and cascade form realizations of

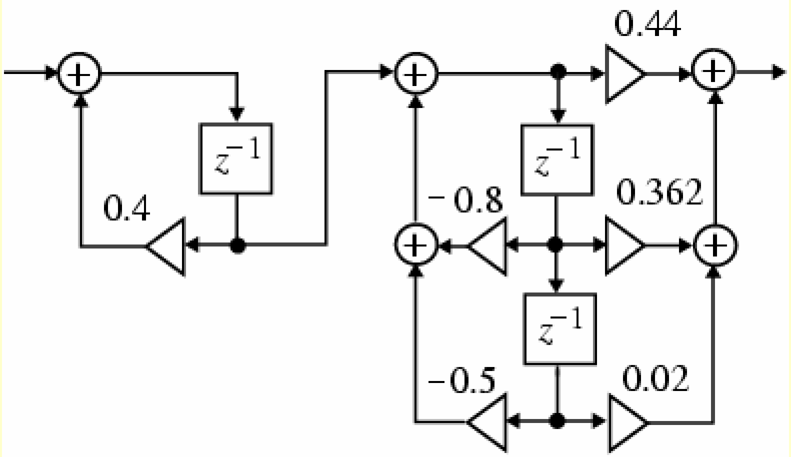
$$H(z)=\frac{0.44z^{-1}+0.362z^{-2}+0.02z^{-3}}{1+0.4z^{-1}+0.18z^{-2}-0.2z^{-3}}$$

$$=\left(\frac{0.44+0.362z^{-1}+0.02z^{-2}}{1+0.8z^{-1}+0.5z^{-2}}\right)\left(\frac{z^{-1}}{1-0.4z^{-1}}\right)$$

are shown on the next slide

Cascade Form IIR Digital Filter Structures





Direct form II

Cascade form

Parallel Form IIR Digital Filter Structures

- A partial-fraction expansion of the transfer function in z^{-1} leads to the **parallel form I** structure
- Assuming simple poles, the transfer function $H(z)$ can be expressed as

$$H(z) = \gamma_0 + \sum_k \left(\frac{\gamma_{0k} + \gamma_{1k} z^{-1}}{1 + \alpha_{1k} z^{-1} + \alpha_{2k} z^{-2}} \right)$$

- In the above for a real pole $\alpha_{2k} = \gamma_{1k} = 0$

Parallel Form IIR Digital Filter Structures

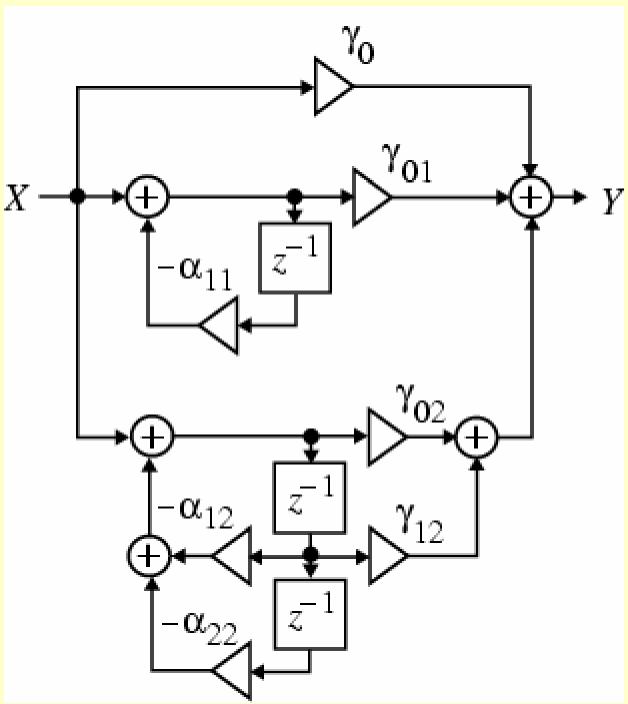
- A direct partial-fraction expansion of the transfer function in z leads to the **parallel form II** structure
- Assuming simple poles, the transfer function $H(z)$ can be expressed as

$$H(z) = \delta_0 + \sum_k \left(\frac{\delta_{1k} z^{-1} + \delta_{2k} z^{-2}}{1 + \alpha_{1k} z^{-1} + \alpha_{2k} z^{-2}} \right)$$

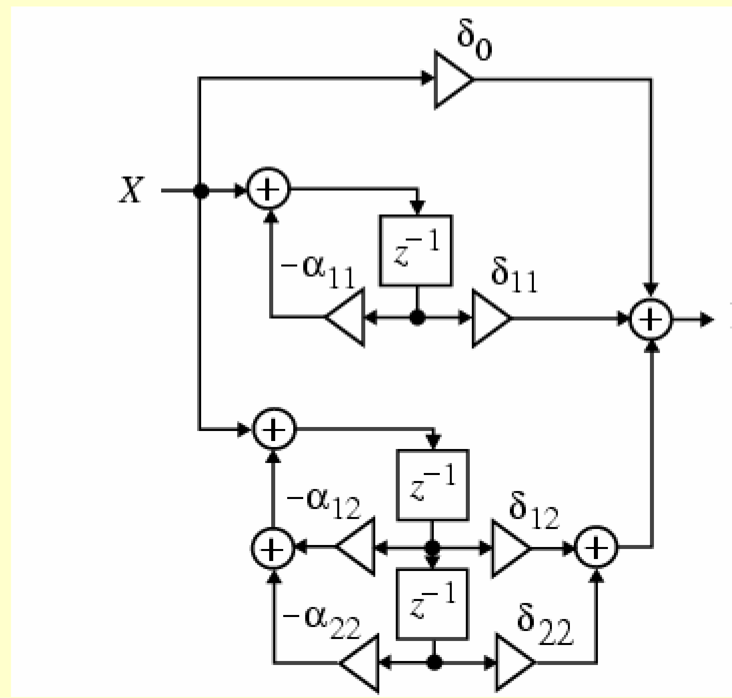
- In the above for a real pole $\alpha_{2k} = \delta_{2k} = 0$

Parallel Form IIR Digital Filter Structures

- The two basic parallel realizations of a 3rd-order IIR transfer function are shown below



Parallel form I



Parallel form II

Parallel Form IIR Digital Filter Structures

- Example - A partial-fraction expansion of

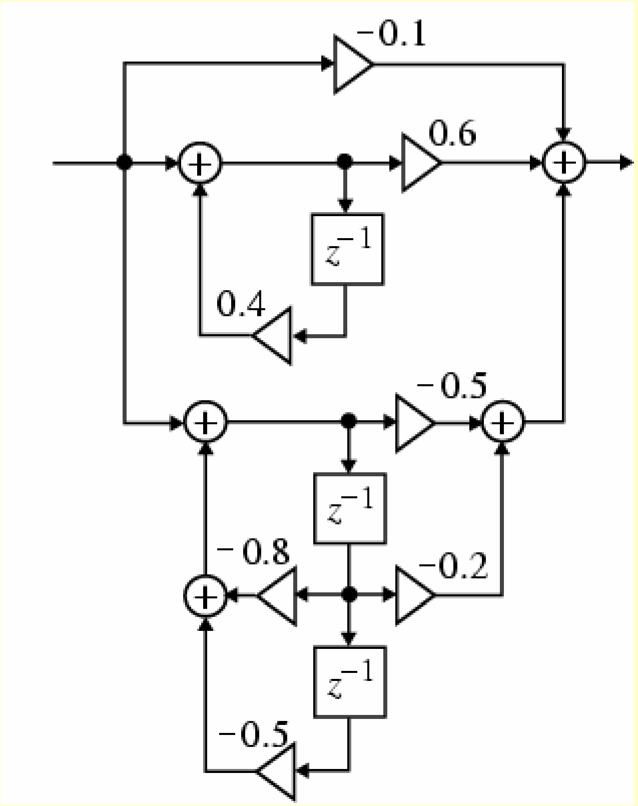
$$H(z)=\frac{0.44z^{-1}+0.362z^{-2}+0.02z^{-3}}{1+0.4z^{-1}+0.18z^{-2}-0.2z^{-3}}$$

in z^{-1} yields

$$H(z)=-0.1+\frac{0.6}{1-0.4z^{-1}}+\frac{-0.5-0.2z^{-1}}{1+0.8z^{-1}+0.5z^{-2}}$$

Parallel Form IIR Digital Filter Structures

- The corresponding parallel form I realization is shown below

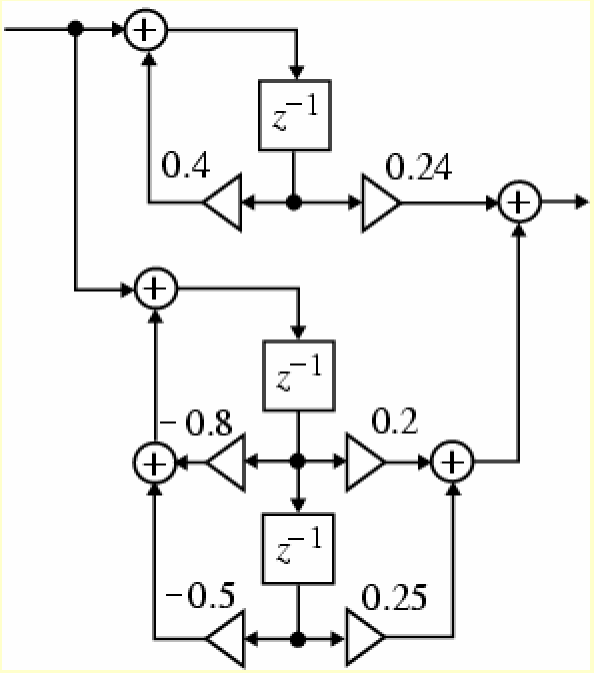


Parallel Form IIR Digital Filter Structures

- Likewise, a partial-fraction expansion of $H(z)$ in z yields

$$H(z) = \frac{0.24z^{-1}}{1-0.4z^{-1}} + \frac{0.2z^{-1}+0.25z^{-2}}{1+0.8z^{-1}+0.5z^{-2}}$$

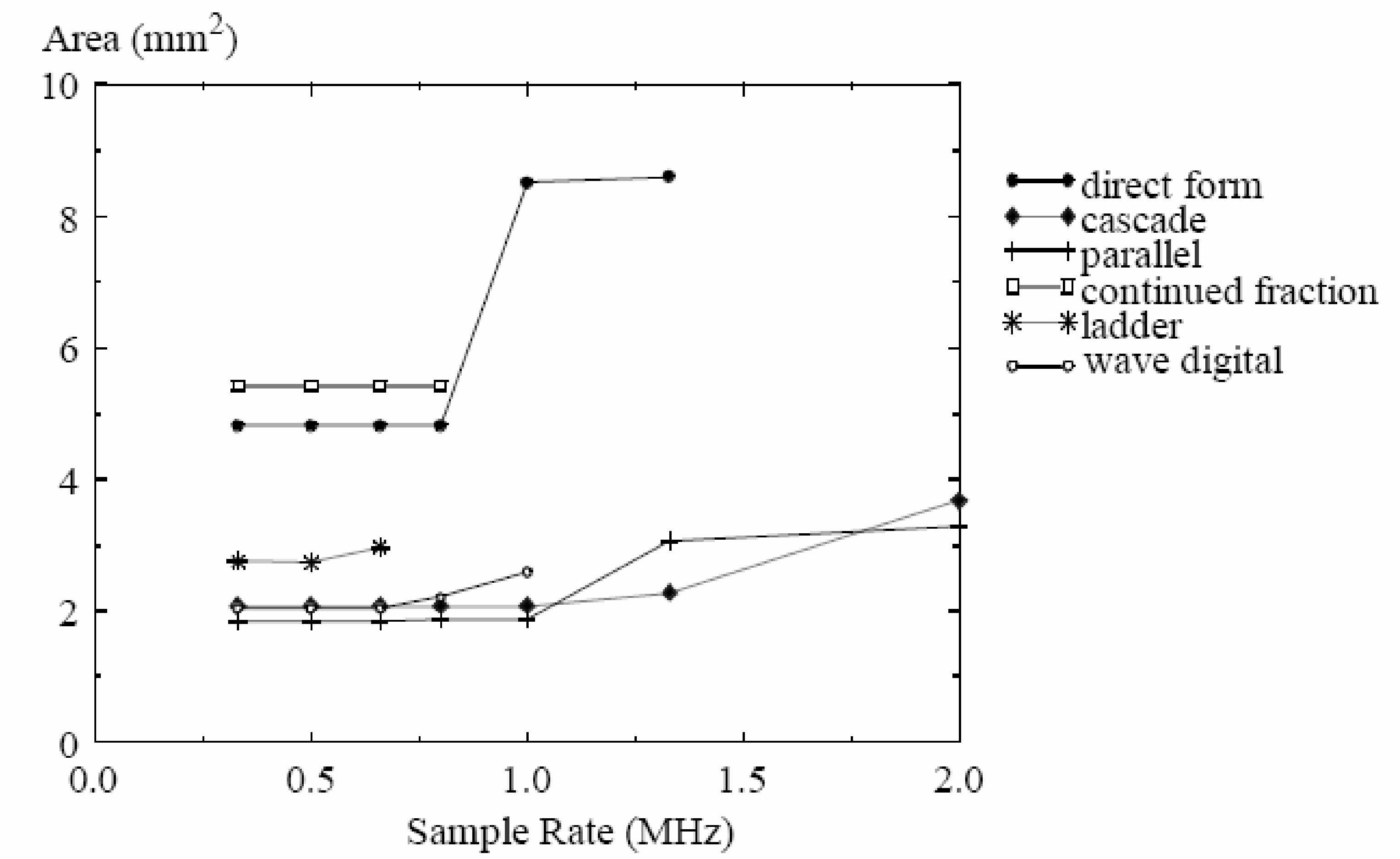
- The corresponding parallel form II realization is shown on the right



Comparison of the complexity of different IIR filters

Structure	Number of multiplications	Number of additions and subtractions	Total number of operations	Required bitwidth
Direct form	16	16	40	21
Cascade	13	16	34	12
Parallel	18	16	39	11
Continued fraction	18	16	35	23
Ladder	17	32	50	14
Wave digital	11	30	47	12

Estimation of area for ASIC implementation



Estimation of number of processors

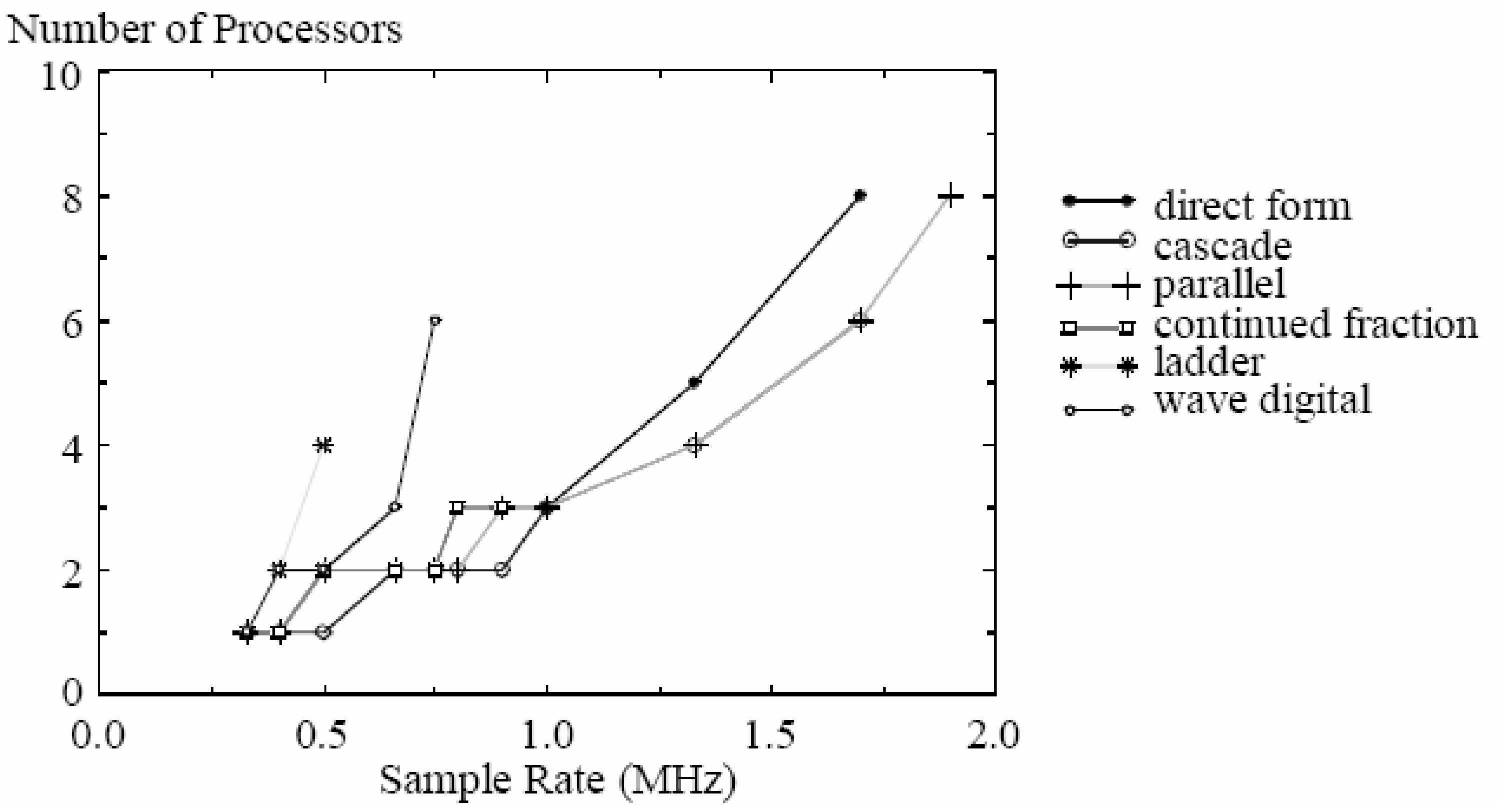


Fig. 32. Number of processors versus sample rate for 24-bit ALU implementations.

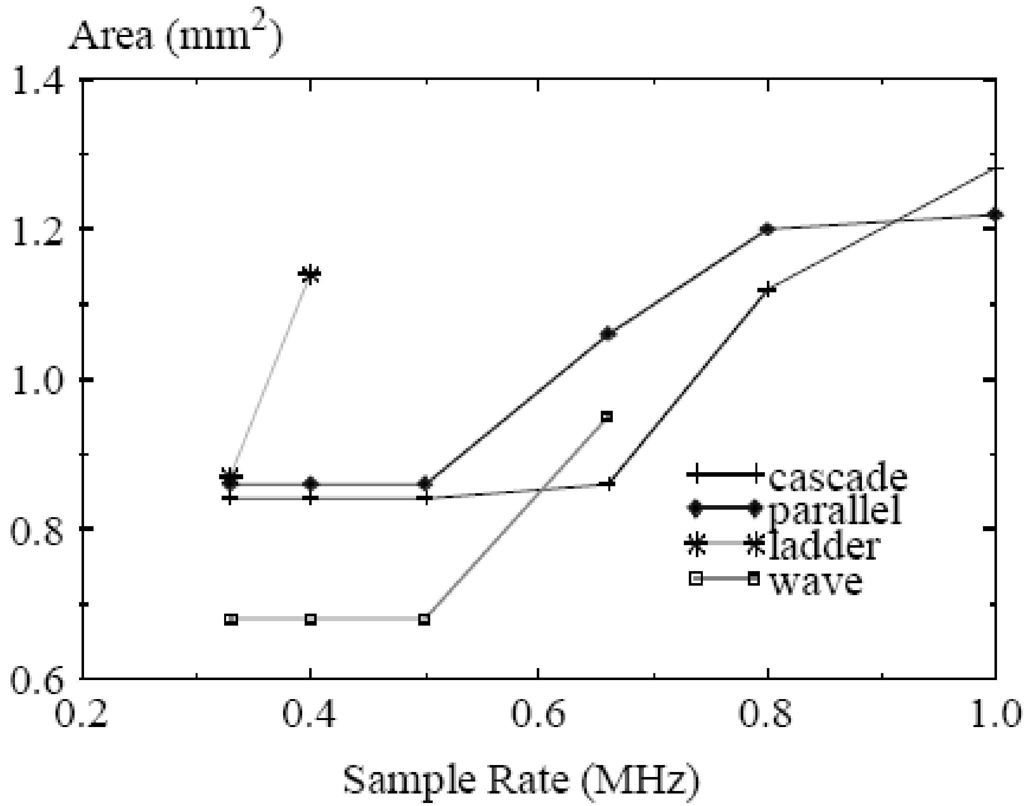


Fig. 33. Area for semi-custom ASIC implementations after constant multiplication expansion into additions and shifts.

Other possibilities for comparisson

- Predicting pipelining improvement using timing metrics.
- Predicting retiming improvement